

Analysis of Discrete Fourier Transform: Finite-Dimensional Hilbert Space in Electroencephalogram Signal

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ABSTRACT

The Discrete Fourier Transform (DFT) allows the digital signals representation in the frequency domain by decomposing discrete signals into finite sinusoidal components. DFT works on discrete data with a certain number of samples relevant for modern digital signal processing. In the context of electroencephalogram (EEG), DFT plays an important role in identifying brain frequency spectra, such as alpha, beta, theta, and delta waves, each of which has physiological and clinical significance. This article analyzes the Discrete Fourier Transform (DFT) in the context of finite-dimensional Hilbert space and its application to analyze EEG signals. The objectives of the study are to prove that the DFT is in Hilbert space by showing its mathematical properties and using DFT to analyze the frequency characteristics of EEG signals in finite-dimensional Hilbert space. The method used involves a mathematical approach by establishing a mathematical framework based on finite-dimensional Hilbert space, analyzing the inverse DFT by recasting the signal from the frequency domain to the time domain, identifying the relevant mathematical characteristics of the DFT. Discretization of continuous signals into discrete ones through impulse sampling techniques, and application of DFT for transformation to the frequency domain for analyzing EEG spectral. The results show that DFT satisfies the definition and axioms of complex vector spaces and finite-dimensional Hilbert spaces, having a provable inverse through the properties of unitary operators. The application of DFT to the modeled EEG signal successfully produces a frequency spectrum that matches the frequency components of the initial model, both in magnitude and energy.



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I. INTRODUCTION

Fourier analysis provides a fundamental framework for representing signals in the frequency domain. For digitally acquired signals, which are represented by finite discrete samples, the Discrete Fourier Transform (DFT) provides a finite-dimensional formulation suitable for numerical computation and digital signal processing. Jean Baptiste Joseph Fourier (1768–1830) was a mathematician who observed complex signals and formulated them as an infinite sum of sinusoidal waves. His findings stated that any signal or wave, no matter how complex, can always be written as a series of simple wave sums [1]. This statement is known as the Fourier series. Periodic functions approximated by partial sums of a Fourier series always converge [2]. Subsequently,

research on Fourier series continued and was developed into a more general form, allowing functions not only to be applied to periodic but also to non-periodic functions [3]. This concept is known as the Fourier transform.

The Fourier transform is one of the most important mathematical tools in science and engineering. Mathematically, the Fourier transform can be viewed as a signal from one domain (usually time) to another, namely the frequency domain. The Fourier transform concept has had a profound impact on the development of science, particularly in fields such as science and engineering. Some of these include audio engineering, which utilizes the ear's sensitivity to frequency as a research object; biomedical engineering, which uses the Fourier transform for health imaging, such as Magnetic Resonance Imaging (MRI), which records

frequency data and converts it to its inverse form to produce an image signal; psychology, which uses the Fourier transform to study different domains of view; brain waves, which use the Fourier transform to view dominant spectra; and others [4].

Initially, the Fourier transform was viewed as a continuous function or analogy signal; however, in computing practice, much data is obtained through digital devices that produce discrete data. The main limitation of the continuous Fourier transform is its inability to handle data that is inherently discrete. For example, an Electroencephalogram (EEG) signal recorded by a computer through human brain activity can only be expressed as a collection of discrete values because computers can only read discrete data. EEG signals are highly nonlinear [5] and nonstationary [6], traditional Fourier analysis which expands the signal in sinusoidal form cannot accurately represent the amplitude contribution of each frequency value [7]. Therefore, a transformation approach must be developed that addresses the characteristics that are lacking in the continuous Fourier transform [8]. This is the reason for the concept called the Discrete Fourier Transform (DFT). Rapid scientific advancements have led to significantly more advanced data processing. Many software and hardware processes data digitally, such as EEG signals, MRI scans, electrocardiograms (ECG), electrooculograms (EOG), and others.

One application of DFT in the neurological field is the analysis of EEG signals. Electroencephalography is a technique for measuring electrical activity in the brain. Electrical activity in the human brain was first introduced by Hans Berger (1924) using a simple galvanometer [9]. Electroencephalography typically measures brain waves, which consist of delta, theta, alpha, beta, and gamma waves. Each type of brain wave has a dominant characteristic in the frequency domain. Delta waves have a frequency of 0.2-3 Hz, which is associated with sleep; theta waves have a frequency of 3-8 Hz, which is dominant in very relaxed states; alpha waves have a frequency of 8-12 Hz, which is usually dominant in relaxed states; beta waves have a frequency of 12-27 Hz, which is typically dominant during focused and alert states; and finally, gamma waves, with a frequency above 27 Hz, are dominant during productive states [10].

The frequency of brain waves is one area of study in DFT because DFT can convert the time domain to the frequency domain. Information about the frequency obtained through DFT will be processed into a frequency spectrum, which represents the signal's distribution against frequency. This frequency spectrum can be described in terms of magnitude (amplitude) and energy (power), which are used to obtain neurological information, including insights into brain diseases. For example, when someone is sleeping, the magnitude and energy of delta waves will be dominant. Still, if in that condition the energy of other waves is tremendous, for example gamma, then this information can be used to provide conclusions about the condition of the brain that has

the potential to experience neurological diseases such as Alzheimer's, Attention Deficit Hyperactivity Disorder (ADHD), insomnia, anxiety disorders, epilepsy, depression, stress, and others [11].

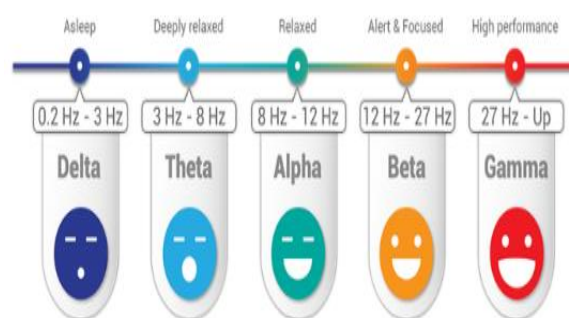


Figure 1. Brain wave frequency and conditions [12]

In this modern era, many data processing devices are digital, so the output data processed through the devices must be discrete. Some examples include electroencephalography, electrocardiography, electrooculography, and even Microsoft Excel, which require discrete input, such as creating a graph. However, the data processed typically originates from functions or continuous data. Therefore, processing data through digital devices requires a concept for transforming continuous data into discrete data. The signals simulated in this study are continuous, but because the DFT operates on discrete signals, another idea is needed to transform continuous signals into discrete ones. A constant signal will be sampled into N samples with each sample point equally spaced, using a distribution called the Dirac delta. The Dirac delta is a distribution function that brings continuous points to discrete samples. This sampling model is known as the impulse sampling technique [13].

Understanding the mathematical structure of the DFT is essential because its basic concepts will provide insight into the space in which the DFT operates. The preliminary assumption regarding the DFT space is based on a finite-dimensional Hilbert space, specifically the vector space C^N . This space has an inner product structure that supports the concepts of orthogonality, norms, and projections [14]. Therefore, this research aims to explore and prove the existence of the DFT within the framework of Hilbert space and investigate other mathematical properties associated with it. The objectives of this research are: (1) to prove that the DFT exists in Hilbert space and demonstrate its mathematical properties. (2) To investigate the frequency characteristics of EEG signals in a finite-dimensional Hilbert space. EEG analysis has become a promising and rapidly growing field of study to understand how the human brain is detected [8]. Theoretically, this research enriches the study of DFT in linear algebra and Fourier analysis, serving as a reference for future theory development. Practically, the study of DFT has numerous applicative benefits, such as in digital signal processing, data compression, image processing, signal reconstruction, and others. It also provides conceptual

references for Fourier analysis, linear algebra, and signal processing. This research can also be further developed in the fields of advanced and applied pure mathematics.

II. METHOD

The research flowchart is presented in Figure 2.

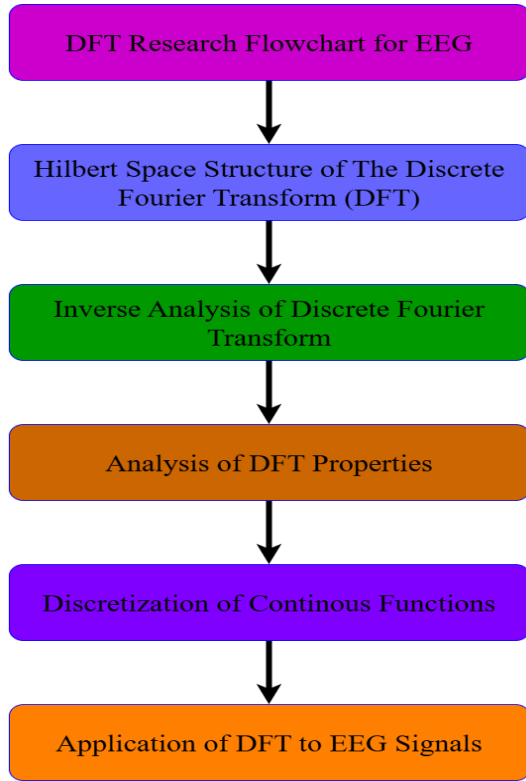


Figure 2. The research procedure

Based on Figure 2, in detail the research was carried out with the following stages:

1. Establishing a mathematical framework based on a finite-dimensional Hilbert space, including: (1) defining EEG signal vectors as elements of the Hilbert space. (2) Determining an orthonormal basis using a complex exponential function. (3) Formulating an inner product to measure signal energy.
2. To analyse the inverse DFT by rearranging the signal from the frequency domain to the time domain in the following order: (1) Writing the inverse DFT formula. (2) Proving the isometry between the time and frequency domains. (3) Testing the consistency of the signal reconstruction results.
3. Identifying relevant mathematical characteristics of the DFT, including: (1) Examining linearity, orthogonality, and periodicity. (2) Demonstrating conservation of energy (Parseval's theorem). (3) Assessing the stability of the transformation against noise.
4. Converting continuous EEG signals into discrete data, including: (1) Determining the sampling frequency according to the Nyquist theorem. (2) Performing amplitude quantization. (3) Constructing a discrete vector representation for analysis. Electroencephalogram (EEG) signal recording is an effective method for determining the brain's electrical activity and learning a lot about how the brain functions [15].
5. Implementing DFT for EEG spectral analysis, including: (1) Retrieving EEG data from a specific channel (2) Applying DFT to obtain a frequency spectrum. (3) Interpreting the results in a neurophysiological context.

III. RESULTS AND DISCUSSION

A. Hilbert Space Structure in The Discrete Fourier Transform (DFT)

Before proving that the DFT is in Hilbert space, it will first be shown that the space defined for the DFT is

$\mathbb{C}^N = \{z = (z_0, z_1, \dots, z_{N-1}) | z_i \in \mathbb{C}, \forall i = 0, 1, 2, \dots, N-1\}$ is a complex vector space. In this case, the proof will be shown as follows. Take any $u, v, w \in \mathbb{C}^N$ with

$$u = \begin{bmatrix} a_0 + jb_0 \\ a_1 + jb_1 \\ \vdots \\ a_{N-1} + jb_{N-1} \end{bmatrix}, v = \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \vdots \\ p_{N-1} + jq_{N-1} \end{bmatrix}, w = \begin{bmatrix} x_0 + jy_0 \\ x_1 + jy_1 \\ \vdots \\ x_{N-1} + jy_{N-1} \end{bmatrix}$$

$a_i, b_i, p_i, q_i, x_i, y_i \in \mathbb{R}, \forall i = 0, 1, 2, \dots, N-1$ and $\mathbf{0}$ is the zero vector and $\lambda, \lambda_1, \lambda_2 \in \mathbb{C}$ scalar. Next, we will examine the ten axioms in the definition of vector space.

a. Closed to addition

$$v + w = \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \vdots \\ p_{N-1} + jq_{N-1} \end{bmatrix} + \begin{bmatrix} x_0 + jy_0 \\ x_1 + jy_1 \\ \vdots \\ x_{N-1} + jy_{N-1} \end{bmatrix} = \begin{bmatrix} p_0 + jq_0 + x_0 + jy_0 \\ p_1 + jq_1 + x_1 + jy_1 \\ \vdots \\ p_{N-1} + jq_{N-1} + x_{N-1} + jy_{N-1} \end{bmatrix} = \begin{bmatrix} (p_0 + x_0) + j(q_0 + y_0) \\ (p_1 + x_1) + j(q_1 + y_1) \\ \vdots \\ (p_N + x_N) + j(q_N + y_N) \end{bmatrix}$$

because $p_i, q_i, x_i, y_i \in \mathbb{R}$, so $p_i + x_i, q_i + y_i \in \mathbb{R}$ so that $v + w \in \mathbb{C}^N, \forall i = 0, 1, \dots, N-1$. Jadi $\mathbb{C}^N$ is the closed to addition.

b. The closed under scalar multiplication

For example $\lambda = r + js$ dengan $r, s \in \mathbb{R}$ so that

$$\lambda u = (r + js) \begin{bmatrix} a_0 + jb_0 \\ a_1 + jb_1 \\ \vdots \\ a_{N-1} + jb_{N-1} \end{bmatrix}$$

$$\lambda u = \begin{bmatrix} (r+js)(a_0+jb_0) \\ (r+js)(a_1+jb_1) \\ \dots \\ (r+js)(a_{N-1}+jb_{N-1}) \\ ra_0+rjb_0+a_0js+j^2sb_0 \\ ra_1+rjb_1+a_1js+j^2sb_1 \\ \dots \\ ra_{N-1}+rjb_{N-1}+a_{N-1}js+j^2sb_{N-1} \end{bmatrix}$$

with $j^2 = -1$

$$\lambda u = \begin{bmatrix} (ra_0-sb_0)+j(rb_0+a_0s) \\ (ra_1-sb_1)+j(rb_1+a_1s) \\ \dots \\ (ra_{N-1}-sb_{N-1})+j(rb_{N-1}+a_{N-1}s) \end{bmatrix}$$

Because $a_i, b_i \in \mathbb{R}$ so $ra_i-sb_i, rb_i+a_is \in \mathbb{R}, \forall i = 0, 1, 2, \dots, N-1$ so that $\lambda u \in \mathbb{C}^N$. Jadi $\mathbb{C}^N$ is the closed under scalar multiplication.

c. The Associative to addition

$$\begin{aligned} u + (v + w) &= \begin{bmatrix} a_0+jb_0 \\ a_1+jb_1 \\ \dots \\ a_{N-1}+jb_{N-1} \end{bmatrix} + \begin{bmatrix} (p_0+x_0)+j(q_0+y_0) \\ (p_1+x_1)+j(q_1+y_1) \\ \dots \\ (p_{N-1}+x_{N-1})+j(q_{N-1}+y_{N-1}) \end{bmatrix} \\ u + (v + w) &= \begin{bmatrix} a_0+jb_0 \\ a_1+jb_1 \\ \dots \\ a_{N-1}+jb_{N-1} \end{bmatrix} + \left(\begin{bmatrix} p_0+jq_0 \\ p_1+jq_1 \\ \dots \\ p_{N-1}+jq_{N-1} \end{bmatrix} + \begin{bmatrix} x_0+jy_0 \\ x_1+jy_1 \\ \dots \\ x_{N-1}+jy_{N-1} \end{bmatrix} \right) \\ u + (v + w) &= \begin{bmatrix} a_0+jb_0+p_0+jq_0 \\ a_1+jb_1+p_1+jq_1 \\ \dots \\ a_{N-1}+jb_{N-1}+p_{N-1}+jq_{N-1} \end{bmatrix} + \begin{bmatrix} x_0+jy_0 \\ x_1+jy_1 \\ \dots \\ x_{N-1}+jy_{N-1} \end{bmatrix} \\ u + (v + w) &= \begin{bmatrix} (a_0+p_0)+j(b_0+q_0) \\ (a_1+p_1)+j(b_1+q_1) \\ \dots \\ (a_{N-1}+p_{N-1})+j(b_{N-1}+q_{N-1}) \end{bmatrix} + \begin{bmatrix} x_0+jy_0 \\ x_1+jy_1 \\ \dots \\ x_{N-1}+jy_{N-1} \end{bmatrix} \\ u + (v + w) &= (u + v) + w \end{aligned}$$

So $\mathbb{C}^N$ is the associative to addition.

d. The Commutative property of addition

$$v + w = \begin{bmatrix} (p_0+x_0)+j(q_0+y_0) \\ (p_1+x_1)+j(q_1+y_1) \\ \dots \\ (p_{N-1}+x_{N-1})+j(q_{N-1}+y_{N-1}) \end{bmatrix}$$

$$\begin{aligned} v + w &= \begin{bmatrix} (x_0+p_0)+j(y_0+q_0) \\ (x_1+p_1)+j(y_1+q_1) \\ \dots \\ (x_{N-1}+p_{N-1})+j(y_{N-1}+q_{N-1}) \end{bmatrix} \\ v + w &= \begin{bmatrix} x_0+jy_0 \\ x_1+jy_1 \\ \dots \\ x_{N-1}+jy_{N-1} \end{bmatrix} + \begin{bmatrix} p_0+jq_0 \\ p_1+jq_1 \\ \dots \\ p_{N-1}+jq_{N-1} \end{bmatrix} \\ v + w &= w + v \end{aligned}$$

So $\mathbb{C}^N$ is the commutative property of addition.

e. Has an identity element of addition

For example

$$\mathbf{0} = \begin{bmatrix} 0+j0 \\ 0+j0 \\ \dots \\ 0+j0 \end{bmatrix} \in \mathbb{C}^N$$

So that

$$\begin{aligned} v + \mathbf{0} &= \begin{bmatrix} p_0+jq_0 \\ p_1+jq_1 \\ \dots \\ p_{N-1}+jq_{N-1} \end{bmatrix} + \begin{bmatrix} 0+j0 \\ 0+j0 \\ \dots \\ 0+j0 \end{bmatrix} \\ v + \mathbf{0} &= \begin{bmatrix} (p_0+0)+j(q_0+0) \\ (p_1+0)+j(q_1+0) \\ \dots \\ (p_{N-1}+0)+j(q_{N-1}+0) \end{bmatrix} \\ v + \mathbf{0} &= \begin{bmatrix} p_0+jq_0 \\ p_1+jq_1 \\ \dots \\ p_{N-1}+jq_{N-1} \end{bmatrix} = v \end{aligned}$$

The other side

$$\begin{aligned} \mathbf{0} + v &= \begin{bmatrix} 0+j0 \\ 0+j0 \\ \dots \\ 0+j0 \end{bmatrix} + \begin{bmatrix} p_0+jq_0 \\ p_1+jq_1 \\ \dots \\ p_{N-1}+jq_{N-1} \end{bmatrix} \\ \mathbf{0} + v &= \begin{bmatrix} (0+p_0)+j(0+q_0) \\ (0+p_1)+j(0+q_1) \\ \dots \\ (0+p_{N-1})+j(0+q_{N-1}) \end{bmatrix} \\ \mathbf{0} + v &= \begin{bmatrix} p_0+jq_0 \\ p_1+jq_1 \\ \dots \\ p_{N-1}+jq_{N-1} \end{bmatrix} = v \end{aligned}$$

So that $\mathbb{C}^N$ has an identity element of addition is $\mathbf{0}$.

f. Having compatibility properties

For example, $\lambda_1 = r + js, \lambda_2 = g + jh$

$$\begin{aligned} \lambda_1(\lambda_2 v) &= \lambda_1 \left((g + jh) \begin{bmatrix} p_0+jq_0 \\ p_1+jq_1 \\ \dots \\ p_{N-1}+jq_{N-1} \end{bmatrix} \right) \\ &= \begin{bmatrix} rp_0g - rq_0h + jrp_0g + jrp_0h + jsp_0g - jsq_0h - sq_0g - sp_0h \\ rp_1g - rq_1h + jrp_1g + jrp_1h + jsp_1g - jsq_1h - sq_1g - sp_1h \\ \dots \\ [rp_{N-1}g - rq_{N-1}h + jrp_{N-1}g + jrp_{N-1}h + jsp_{N-1}g - jsq_{N-1}h - sq_{N-1}g - sp_{N-1}h] \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} (rp_0g - rq_0h - sq_0g - sp_0h) + j(rq_0g + rp_0h + sp_0g - sq_0h) \\ (rp_1g - rq_1h - sq_1g - sp_1h) + j(rq_1g + rp_1h + sp_1g - sq_1h) \\ \dots \\ [(rp_{N-1}g - rq_{N-1}h - sq_{N-1}g - sp_{N-1}h) + \\ j(rq_{N-1}g + rp_{N-1}h + sp_{N-1}g - sq_{N-1}h)] \end{bmatrix}$$

The other side

$$(\lambda_1 \lambda_2) \mathbf{v} = ((r + js)(g + jh))$$

$$(\lambda_1 \lambda_2) \mathbf{v} = ((rg - sh) + j(rh + sg)) \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} [p_0rg - p_0sh - q_0rh - q_0sg] + j[p_0rh + p_0sg + q_0rg - q_0sh] \\ [p_1rg - p_1sh - q_1rh - q_1sg] + j[p_1rh + p_1sg + q_1rg - q_1sh] \\ \dots \\ [p_{N-1}rg - p_{N-1}sh - q_{N-1}rh - q_{N-1}sg] + j[p_{N-1}rh + p_{N-1}sg + q_{N-1}rg - q_{N-1}sh] \end{bmatrix}$$

So that $(\lambda_1 \lambda_2) \mathbf{v} = \lambda_1 (\lambda_2 \mathbf{v})$. Its mean, to has compatibility properties

g. Has a multiplicative identity

For example, one (1) in the context of complex numbers is defined

$$1 = 1 + j0 \in \mathbb{C}$$

So that

$$1\mathbf{v} = (1 + j0) \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1.p_0 - q_00 + j(1.q_0 + p_00) \\ 1.p_1 - q_10 + j(1.q_1 + p_10) \\ \dots \\ 1.p_{N-1} - q_{N-1}0 + j(1.q_{N-1} + p_{N-1}0) \end{bmatrix}$$

$$= \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix} = \mathbf{v}$$

h. Having zero properties

For example, $0 = 0 + j0 \in \mathbb{C}$

$$0\mathbf{v} = (0 + j0) \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0.p_0 - q_00 + j(0.q_0 + p_00) \\ 0.p_1 - q_10 + j(0.q_1 + p_10) \\ \dots \\ 0.p_{N-1} - q_{N-1}0 + j(0.q_{N-1} + p_{N-1}0) \end{bmatrix}$$

$$0\mathbf{v} = (0 + j0) \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix} = \begin{bmatrix} 0 + j0 \\ 0 + j0 \\ \dots \\ 0 + j0 \end{bmatrix} = \mathbf{0}$$

i. Having Distributive vector

$$\lambda(\mathbf{v} + \mathbf{w}) = \lambda \left(\begin{bmatrix} (p_0 + x_0) + j(q_0 + y_0) \\ (p_1 + x_1) + j(q_1 + y_1) \\ \dots \\ (p_{N-1} + x_{N-1}) + j(q_{N-1} + y_{N-1}) \end{bmatrix} \right)$$

$$\lambda(\mathbf{v} + \mathbf{w}) = \lambda \begin{bmatrix} p_0 + jq_0 + x_0 + jy_0 \\ p_1 + jq_1 + x_1 + jy_1 \\ \dots \\ p_{N-1} + jq_{N-1} + x_{N-1} + jy_{N-1} \end{bmatrix}$$

$$\lambda(\mathbf{v} + \mathbf{w}) = \begin{bmatrix} \lambda p_0 + j\lambda q_0 + \lambda x_0 + j\lambda y_0 \\ \lambda p_1 + j\lambda q_1 + \lambda x_1 + j\lambda y_1 \\ \dots \\ \lambda p_{N-1} + j\lambda q_{N-1} + \lambda x_{N-1} + j\lambda y_{N-1} \end{bmatrix}$$

$$\lambda(\mathbf{v} + \mathbf{w}) = \begin{bmatrix} (\lambda p_0 + j\lambda q_0) + (\lambda x_0 + j\lambda y_0) \\ (\lambda p_1 + j\lambda q_1) + (\lambda x_1 + j\lambda y_1) \\ \dots \\ (\lambda p_{N-1} + j\lambda q_{N-1}) + (\lambda x_{N-1} + j\lambda y_{N-1}) \end{bmatrix}$$

$$\lambda(\mathbf{v} + \mathbf{w}) = \begin{bmatrix} \lambda p_0 + j\lambda q_0 \\ \lambda p_1 + j\lambda q_1 \\ \dots \\ \lambda p_{N-1} + j\lambda q_{N-1} \end{bmatrix} + \begin{bmatrix} \lambda x_0 + j\lambda y_0 \\ \lambda x_1 + j\lambda y_1 \\ \dots \\ \lambda x_{N-1} + j\lambda y_{N-1} \end{bmatrix}$$

$$\lambda(\mathbf{v} + \mathbf{w}) = \lambda \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix} + \lambda \begin{bmatrix} x_0 + jy_0 \\ x_1 + jy_1 \\ \dots \\ x_{N-1} + jy_{N-1} \end{bmatrix}$$

$$\lambda(\mathbf{v} + \mathbf{w}) = \lambda \mathbf{v} + \lambda \mathbf{w}$$

j. Has scalar distributive

$$(\lambda_1 + \lambda_2) \mathbf{v} = (\lambda_1 + \lambda_2) \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda_1 + \lambda_2)p_0 + (\lambda_1 + \lambda_2)jq_0 \\ (\lambda_1 + \lambda_2)p_1 + (\lambda_1 + \lambda_2)jq_1 \\ \dots \\ (\lambda_1 + \lambda_2)p_{N-1} + (\lambda_1 + \lambda_2)jq_{N-1} \end{bmatrix}$$

$$(\lambda_1 + \lambda_2) \mathbf{v} = \begin{bmatrix} p_0\lambda_1 + p_0\lambda_2 + j(q_0\lambda_1 + q_0\lambda_2) \\ p_1\lambda_1 + p_1\lambda_2 + j(q_1\lambda_1 + q_1\lambda_2) \\ \dots \\ p_{N-1}\lambda_1 + p_{N-1}\lambda_2 + j(q_{N-1}\lambda_1 + q_{N-1}\lambda_2) \end{bmatrix}$$

$$= \begin{bmatrix} (p_0\lambda_1 + jq_0\lambda_1) + (p_0\lambda_2 + jq_0\lambda_2) \\ (p_1\lambda_1 + jq_1\lambda_1) + (p_1\lambda_2 + jq_1\lambda_2) \\ \dots \\ (p_{N-1}\lambda_1 + jq_{N-1}\lambda_1) + (p_{N-1}\lambda_2 + jq_{N-1}\lambda_2) \end{bmatrix}$$

$$(\lambda_1 + \lambda_2) \mathbf{v} = \begin{bmatrix} \lambda_1(p_0 + jq_0) + \lambda_2(p_0 + jq_0) \\ \lambda_1(p_1 + jq_1) + \lambda_2(p_1 + jq_1) \\ \dots \\ \lambda_1(p_{N-1} + jq_{N-1}) + \lambda_2(p_{N-1} + jq_{N-1}) \end{bmatrix}$$

$$(\lambda_1 + \lambda_2) \mathbf{v} = \lambda_1 \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix} + \lambda_2 \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix}$$

$$(\lambda_1 + \lambda_2) \mathbf{v} = \lambda_1 \mathbf{v} + \lambda_2 \mathbf{v}$$

Thus, the space defined in the DFT, $\mathbb{C}^N$, is a complex vector space under the operations of addition and scalar multiplications. Furthermore, suppose $\mathbb{C}^N$ is a complex vector space with N components equipped with an inner product defined as

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{k=0}^{N-1} a_k b_k^*$$

With $\mathbf{a}, \mathbf{b} \in \mathbb{C}^N$ two vectors with N complex components. Then, we will show that $\mathbb{C}^N$, with an inner product, is an inner product space. The proof is as follows. Take any

$$\mathbf{a} = \begin{bmatrix} x_0 + jy_0 \\ x_1 + jy_1 \\ \dots \\ x_{N-1} + jy_{N-1} \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} p_0 + jq_0 \\ p_1 + jq_1 \\ \dots \\ p_{N-1} + jq_{N-1} \end{bmatrix}$$

which are elements of the vector space $\mathbb{C}^N$ such that

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{k=0}^{N-1} a_k b_k^*$$

with b_k^* is a complex conjugate b_k , $\forall k = 0, 1, 2, \dots, N-1$,

$$b_k^* = p_k - jq_k$$

So that

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{k=0}^{N-1} (x_k + jy_k)(p_k - jq_k)$$

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{k=0}^{N-1} x_k p_k - jx_k q_k + jy_k p_k - j^2 y_k q_k$$

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{k=0}^{N-1} x_k p_k - jx_k q_k + jy_k p_k + y_k q_k, j^2 = -1$$

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{k=0}^{N-1} (x_k p_k + y_k q_k) + j(-x_k q_k + y_k p_k)$$

The other side

$$\langle \mathbf{b}, \mathbf{a} \rangle^* = \left(\sum_{k=0}^{N-1} b_k a_k^* \right)^*$$

with $a_k^* = x_k - jy_k$ so that got,

$$\langle \mathbf{b}, \mathbf{a} \rangle^* = \left(\sum_{k=0}^{N-1} (p_k + jq_k)(x_k - jy_k) \right)^*$$

$$\langle \mathbf{b}, \mathbf{a} \rangle^* = \left(\sum_{k=0}^{N-1} p_k x_k - jp_k y_k + jq_k x_k - j^2 q_k y_k \right)^*$$

$$\langle \mathbf{b}, \mathbf{a} \rangle^* = \left(\sum_{k=0}^{N-1} p_k x_k - jp_k y_k + jq_k x_k + q_k y_k \right)^*, j^2 = -1$$

$$\langle \mathbf{b}, \mathbf{a} \rangle^* = \left(\sum_{k=0}^{N-1} (p_k x_k + q_k y_k) + j(-p_k y_k + q_k x_k) \right)^*$$

Based on the theorem,

$$\operatorname{Re}(S^*) = \sum_{m=1}^N x_m \text{ dan } \operatorname{Im}(S^*) = -\sum_{m=1}^N y_m \text{ if } S^* \text{ conjugate of } S.$$

Then it is obtained

$$\langle \mathbf{b}, \mathbf{a} \rangle^* = \sum_{k=0}^{N-1} (p_k x_k + q_k y_k) - j(-p_k y_k + q_k x_k)$$

$$\langle \mathbf{b}, \mathbf{a} \rangle^* = \sum_{k=0}^{N-1} (p_k x_k + q_k y_k) + j(p_k y_k - q_k x_k)$$

$$\langle \mathbf{b}, \mathbf{a} \rangle^* = \sum_{k=0}^{N-1} (x_k p_k + y_k q_k) + j(-x_k q_k + y_k p_k) = \langle \mathbf{a}, \mathbf{b} \rangle$$

Next, take any complex vector

$$\mathbf{c} = \begin{bmatrix} r_0 + js_0 \\ r_1 + js_1 \\ \dots \\ r_{N-1} + js_{N-1} \end{bmatrix} \in \mathbb{C}^N$$

$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = \sum_{k=0}^{N-1} (a_k + b_k) c_k^*$$

$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = \sum_{k=0}^{N-1} (a_k c_k^* + b_k c_k^*)$$

$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = \sum_{k=0}^{N-1} a_k c_k^* + \sum_{k=0}^{N-1} b_k c_k^*$$

$$\langle \mathbf{a} + \mathbf{b}, \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle + \langle \mathbf{b}, \mathbf{c} \rangle$$

Then, for α , the complex scalar

$$\langle \alpha \mathbf{a}, \mathbf{b} \rangle = \sum_{k=0}^{N-1} (\alpha a_k) b_k^* = \alpha \sum_{k=0}^{N-1} a_k b_k^* = \alpha \langle \mathbf{a}, \mathbf{b} \rangle$$

Next, its positive-definite nature will be demonstrated.

$$\langle \mathbf{a}, \mathbf{a} \rangle = \sum_{k=0}^{N-1} a_k a_k^* = \sum_{k=0}^{N-1} (x_k + jy_k)(x_k - jy_k)$$

$$\langle \mathbf{a}, \mathbf{a} \rangle = \sum_{k=0}^{N-1} x_k^2 - j^2 y_k^2 = \sum_{k=0}^{N-1} x_k^2 + y_k^2$$

Because x_k, y_k for each $k = 0, 1, 2, \dots, N-1$ has real value then x_k^2, y_k^2 non-negative real so that

$$\langle \mathbf{a}, \mathbf{a} \rangle > 0$$

for $\mathbf{a} = \mathbf{0} \in \mathbb{C}^N$

$$\langle \mathbf{0}, \mathbf{0} \rangle = \sum_{k=0}^{N-1} 0_k^2 + 0_k^2 = 0$$

So that $\langle \mathbf{a} \rangle = 0$ if $\mathbf{a} \in \mathbb{C}^N$. Thus, $\mathbb{C}^N$, equipped with an inner product, is an inner product space.

Since $\mathbb{C}^N$ fitted with an inner product is an inner product space, it has been proven that $\mathbb{C}^N$ is also a normed space because, based on the definition of an inner product space, each element in the vector space $\mathbb{C}^N$ has a length or norm, namely

$$\begin{aligned} \|\mathbf{a}\| &= \langle \mathbf{a}, \mathbf{a} \rangle^{\frac{1}{2}} = \left(\sum_{k=0}^{N-1} (x_k + jy_k)(x_k - jy_k) \right)^{\frac{1}{2}} \\ &= \left(\sum_{k=0}^{N-1} x_k^2 + y_k^2 \right)^{\frac{1}{2}} < \infty \end{aligned}$$

Then, to prove that $\mathbb{C}^N$ is a Hilbert space, it must first be shown that $\mathbb{C}^N$ is a complete metric space. The proof is as follows. Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^N$ complex vector of N components and

$$x_k = a_k + jb_k \in \mathbf{x}, y_k = c_k + jd_k \in \mathbf{y}$$

$$z_k = e_k + f_k \in \mathbf{z} \quad \forall k = 0, 1, 2, \dots, N-1$$

So that

$$\mathbf{x} = \begin{bmatrix} a_0 + jb_0 \\ a_1 + jb_1 \\ \dots \\ a_{N-1} + jb_{N-1} \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} c_0 + jd_0 \\ c_1 + jd_1 \\ \dots \\ c_{N-1} + jd_{N-1} \end{bmatrix}$$

$$\mathbf{z} = \begin{bmatrix} e_0 + jf_0 \\ e_1 + jf_1 \\ \dots \\ e_{N-1} + jf_{N-1} \end{bmatrix}$$

With using definition: A metric space is a pair (V, d) , where V is a set and d is a metric on V (the distance of a function on V), a function defined on $V \times V$ such that for all $x, y, z \in V$, it must satisfy

(a) Clearly, d is real, finite, and nonnegative, as seen in the proof of the norm space above and in the positive definite evidence in the proof of the inner product space.

(b) $\Rightarrow d(\mathbf{x}, \mathbf{y}) = 0$

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle^{\frac{1}{2}} = \left(\sum_{k=0}^{N-1} (x_k - y_k)(x_k - y_k)^* \right)^{\frac{1}{2}}$$

$$= \left(\sum_{k=0}^{N-1} ((a_k - c_k)^2 - j^2(b_k - d_k)^2) \right)^{\frac{1}{2}}$$

$$= \left(\sum_{k=0}^{N-1} ((- (c_k - a_k))^2 - j^2(-(d_k - b_k))^2) \right)^{\frac{1}{2}}$$

$$d(\mathbf{x}, \mathbf{y}) = \left(\sum_{k=0}^{N-1} ((c_k - a_k)^2 - j^2(d_k - b_k)^2) \right)^{1/2}$$

$$d(\mathbf{x}, \mathbf{y}) = \left(\sum_{k=0}^{N-1} (a_k - c_k)^2 + (b_k - d_k)^2 \right)^{\frac{1}{2}} = 0$$

The equation is fulfilled if

$$(a_k - c_k)^2 = 0$$

dan

$$(b_k - d_k)^2 = 0, \forall k = 0, 1, 2, \dots, N-1.$$

So $a_k = c_k$ dan $b_k = d_k$, so that

$$x_k = y_k, \forall k = 0, 1, 2, \dots, N-1,$$

the consequences $\mathbf{x} = \mathbf{y}$.

$\Leftarrow \mathbf{x} = \mathbf{y}$

Because $\mathbf{x} = \mathbf{y}$, maka $x_k = y_k$ the consequences $a_k = c_k$ and $b_k = d_k$ for each $k = 0, 1, 2, \dots, N-1$ so that

$$d(\mathbf{x}, \mathbf{y}) = \left(\sum_{k=0}^{N-1} (a_k - c_k)^2 + (b_k - d_k)^2 \right)^{\frac{1}{2}}$$

$$d(\mathbf{x}, \mathbf{y}) = \left(\sum_{k=0}^{N-1} (0)^2 + (0)^2 \right)^{\frac{1}{2}} = \left(\sum_{k=0}^{N-1} 0 \right)^{\frac{1}{2}} = 0$$

So it is proven that $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$

$$(c) \quad d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle^{\frac{1}{2}}$$

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle^{\frac{1}{2}} = \left(\sum_{k=0}^{N-1} (x_k - y_k)(x_k - y_k)^* \right)^{\frac{1}{2}}$$

$$= \left(\sum_{k=0}^{N-1} ((a_k - c_k)^2 - j^2(b_k - d_k)^2) \right)^{\frac{1}{2}}$$

$$= \left(\sum_{k=0}^{N-1} ((- (c_k - a_k))^2 - j^2(-(d_k - b_k))^2) \right)^{\frac{1}{2}}$$

$$= \left(\sum_{k=0}^{N-1} ((c_k - a_k)^2 - j^2(d_k - b_k)^2) \right)^{1/2}$$

$$= \left(\sum_{k=0}^{N-1} (y_k - x_k)((c_k + jd_k) - (a_k + jb_k))^* \right)^{\frac{1}{2}}$$

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle^{\frac{1}{2}} = \left(\sum_{k=0}^{N-1} (y_k - x_k)(y_k - x_k)^* \right)^{\frac{1}{2}}$$

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle^{\frac{1}{2}} = \langle \mathbf{y} - \mathbf{x}, \mathbf{y} - \mathbf{x} \rangle^{\frac{1}{2}}$$

$$\left(\sum_{k=0}^{N-1} (y_k - x_k)(y_k - x_k)^* \right)^{\frac{1}{2}} = \langle \mathbf{y} - \mathbf{x}, \mathbf{y} - \mathbf{x} \rangle^{\frac{1}{2}}$$

So that

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle^{\frac{1}{2}} = \langle \mathbf{y} - \mathbf{x}, \mathbf{y} - \mathbf{x} \rangle^{\frac{1}{2}}$$

(d) We will prove the triangle inequality.

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$

Note that $\mathbf{x} - \mathbf{y} = (\mathbf{x} - \mathbf{z}) + (\mathbf{z} - \mathbf{y})$ so that

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{z} + (\mathbf{z} - \mathbf{y})\|$$

By using the vector norm inequality

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{z} + (\mathbf{z} - \mathbf{y})\|$$

$$d(\mathbf{x}, \mathbf{y}) \leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\|$$

$$d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$$

Next, we will demonstrate that the vector space $\mathbb{C}^N$ is a complete metric space, meaning that every Cauchy sequence converges in $\mathbb{C}^N$. Let $\{x^{(n)}\}$ be any Cauchy sequence on $\mathbb{C}^N$ defined as follows.

$$x^{(1)} = (x_0^{(1)}, x_1^{(1)}, x_2^{(1)}, \dots, x_{N-1}^{(1)})$$

$$x^{(2)} = (x_0^{(2)}, x_1^{(2)}, x_2^{(2)}, \dots, x_{N-1}^{(2)})$$

...

$$x^{(n)} = (x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \dots, x_{N-1}^{(n)})$$

is a Cauchy sequence meaning that for every $\varepsilon > 0$, there is $n_0 \in N$ such that for every $n, m \geq n_0$ it holds $d(x^{(n)}, x^{(m)}) < \varepsilon$, take any $\varepsilon > 0$, such that,

$$\begin{aligned} d(x^{(n)}, x^{(m)}) &= \|x^{(n)} - x^{(m)}\| \\ &= \langle x^{(n)} - x^{(m)}, x^{(n)} - x^{(m)} \rangle^{\frac{1}{2}} \\ d(x^{(n)}, x^{(m)}) &= \left(\sum_{k=0}^{N-1} (x_k^{(n)} - x_k^{(m)})(x_k^{(n)} - x_k^{(m)})^* \right)^{\frac{1}{2}} \\ d(x^{(n)}, x^{(m)})^2 &= \sum_{k=0}^{N-1} |x_k^{(n)} - x_k^{(m)}|^2 \end{aligned}$$

Note that $x_k^{(n)}$ is the to- k component of the sequence n such that

$$\begin{aligned} |x_k^{(n)} - x_k^{(m)}|^2 &\leq \sum_{k=0}^{N-1} |x_k^{(n)} - x_k^{(m)}|^2 = d(x^{(n)}, x^{(m)})^2 \\ |x_k^{(n)} - x_k^{(m)}|^2 &< \varepsilon^2 \\ |x_k^{(n)} - x_k^{(m)}| &< \varepsilon \end{aligned}$$

This means that every component of the sequence $\{x^{(n)}\}$ in the complex field is also a Cauchy sequence. Then, since the sequence $\{x^{(n)}\}$ is in the complex vector space $\mathbb{C}^N$ with complex N , every component of the sequence $\{x^{(n)}\}$, namely $x_k^{(n)}$, will converge to one of the components of the sequence in $\mathbb{C}^N$. So $\mathbb{C}^N$ is a complete metric space. So that V equipped with an inner product is a Hilbert space.

B. DFT inverse

After knowing that $\mathbb{C}^N$ is in Hilbert space, we will next show that $\mathcal{F}: \mathbb{C}^N \rightarrow \mathbb{C}^N$, where $\mathcal{F}$ is a unitary operator. First, we will prove that $\mathcal{F}$ is a linear operator. Take any $\mathbf{a}, \mathbf{b} \in \mathbb{C}^N$ with

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \dots \\ a_{N-1} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_{N-1} \end{bmatrix}$$

such that by using equation (2.3)

$$F_k \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}}.$$

The following form is obtained,

$$\mathcal{F}(\mathbf{a} + \mathbf{b}) = \sum_{n=0}^{N-1} (a_n + b_n) e^{\frac{j2\pi nk}{N}}$$

$$\mathcal{F}(\mathbf{a} + \mathbf{b}) = \sum_{n=0}^{N-1} a_n e^{\frac{j2\pi nk}{N}} + \sum_{n=0}^{N-1} b_n e^{\frac{j2\pi nk}{N}} = \mathcal{F}(\mathbf{a}) + \mathcal{F}(\mathbf{b})$$

Then for a scalar c

$$\mathcal{F}(c\mathbf{a}) = \sum_{n=0}^{N-1} (ca_n) e^{\frac{j2\pi nk}{N}} = c \sum_{n=0}^{N-1} a_n e^{\frac{j2\pi nk}{N}} = c\mathcal{F}(\mathbf{a})$$

Since $\mathcal{F}$ is a linear transformation and the domain and codomain are in the same space, namely $\mathbb{C}^N$, $\mathcal{F}$ is also a linear operator. Next, we will prove that $\mathcal{F}$ is bounded using Definition-1.

Definition-1:

Suppose X and Y are normed spaces. A linear operator $T: X \rightarrow Y$ is called a bounded linear operator if there exists a positive constant $c > 0$ such that

$$\|T(x)\|_Y \leq c\|x\|_X$$

Previously, it was proven that $\mathbb{C}^N$ is a normed space. The proof will be outlined as follows.

$$\|\mathcal{F}(\mathbf{f})\|_{\mathbb{C}^N} = \langle \mathcal{F}(\mathbf{f}), \mathcal{F}(\mathbf{f}) \rangle^{\frac{1}{2}} = \langle \mathbf{F}, \mathbf{F} \rangle^{\frac{1}{2}}$$

$$\|\mathcal{F}(\mathbf{f})\|_{\mathbb{C}^N} = \sqrt{\sum_{m=0}^{N-1} F_m \cdot F_m^*}$$

$$\|\mathcal{F}(\mathbf{f})\|_{\mathbb{C}^N} = \sqrt{\sum_{m=0}^{N-1} |F_m|^2}$$

$$\begin{aligned} &\sqrt{\sum_{m=0}^{N-1} \left| \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nm}{N}} \right|^2} = \sqrt{\sum_{m=0}^{N-1} \left(\left| \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nm}{N}} \right| \right)^2} \\ &\leq \sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |f_n e^{-\frac{j2\pi nm}{N}}| \right)^2} = \sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |f_n| \left| e^{-\frac{j2\pi nm}{N}} \right| \right)^2} \end{aligned}$$

Because

$$\left| e^{-\frac{j2\pi nm}{N}} \right| = \left| \cos\left(-\frac{2\pi nm}{N}\right) + j \sin\left(-\frac{2\pi nm}{N}\right) \right| = 1$$

So that

$$\sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |f_n e^{-\frac{j2\pi nm}{N}}| \right)^2} = \sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |f_n| \right)^2}$$

$$\sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |f_n e^{-\frac{j2\pi nm}{N}}| \right)^2} = \sqrt{\left(\sum_{n=0}^{N-1} |f_n| \right)^2 \sum_{m=1}^{N-1} 1}$$

$$\sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |f_n e^{-\frac{j2\pi nm}{N}}| \right)^2} = \sqrt{\left(\sum_{n=0}^{N-1} |f_n| \right)^2 N}$$

$$\sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |f_n e^{-\frac{j2\pi nm}{N}}| \right)^2} = \sqrt{N \left(\sum_{n=0}^{N-1} |f_n| \cdot 1 \right)^2}$$

$$\sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} |f_n e^{-\frac{j2\pi nm}{N}}| \right)^2} \leq \sqrt{N \left(\sum_{n=0}^{N-1} |f_n|^2 \right) \left(\sum_{n=0}^{N-1} 1^2 \right)}$$

$$\sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} \left| f_n e^{-\frac{j2\pi nm}{N}} \right|^2 \right)} = \sqrt{N^2 \left(\sum_{n=0}^{N-1} |f_n|^2 \right)}$$

$$\sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} \left| f_n e^{-\frac{j2\pi nm}{N}} \right|^2 \right)} = N \sqrt{\sum_{n=0}^{N-1} f_n \cdot f_n^*} = N \|f\|_{\mathbb{C}^N}^{\frac{1}{2}}$$

$$\sqrt{\sum_{m=0}^{N-1} \left(\sum_{n=0}^{N-1} \left| f_n e^{-\frac{j2\pi nm}{N}} \right|^2 \right)} = N \|f\|_{\mathbb{C}^N}$$

So that $\|\mathcal{F}(f)\|_{\mathbb{C}^N}^2 \leq N \|f\|_{\mathbb{C}^N}^2$ as a result there is $c = N > 0$ so $\|\mathcal{F}(f)\|_{\mathbb{C}^N}^2 \leq N \|f\|_{\mathbb{C}^N}^2$. So that $\mathcal{F}$ is a bounded linear operator. Finally, it will be shown that $\mathcal{F}$ is unitary. Another representation of a linear operator can be using a transformation matrix of the form $F = \mathcal{A}f$ or $F = \mathcal{F}_A(f)$.

Definition-2

The normalized DFT is defined as

$$F_k \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}}$$

the normalized form of the DFT is

$$\mathcal{F}(f) \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}}$$

$$\mathcal{F}(f) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} \left(\frac{1}{\sqrt{N}} f_n e^{-\frac{j2\pi nk}{N}} \right)$$

$$F_k = \mathcal{F}(f) \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} \left(\frac{1}{\sqrt{N}} e^{-\frac{j2\pi nk}{N}} f_n \right)$$

$$F_k = \mathcal{F}(f) = \frac{1}{\sqrt{N}} e^{-\frac{j2\pi 0k}{N}} f_0 + \frac{1}{\sqrt{N}} e^{-\frac{j2\pi 1k}{N}} f_1 + \dots$$

$$+ \frac{1}{\sqrt{N}} e^{-\frac{j2\pi (N-1)k}{N}} f_{N-1}$$

Because $F \in \mathbb{C}^N$ so $F = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_k \\ \vdots \\ F_{N-1} \end{bmatrix}$ with $k = 0, 1, 2, \dots, N-1$. So

that

$$\mathcal{F}_A(f) = F = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{N-1} \end{bmatrix}$$

$$= \frac{1}{\sqrt{N}} \begin{bmatrix} e^{-\frac{j2\pi(0)(0)}{N}} & e^{-\frac{j2\pi(1)(0)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(0)}{N}} \\ e^{-\frac{j2\pi(0)(1)}{N}} & e^{-\frac{j2\pi(1)(1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{j2\pi(0)(N-1)}{N}} & e^{-\frac{j2\pi(1)(N-1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

With

$$A = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{-\frac{j2\pi(0)(0)}{N}} & e^{-\frac{j2\pi(1)(0)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(0)}{N}} \\ e^{-\frac{j2\pi(0)(1)}{N}} & e^{-\frac{j2\pi(1)(1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{j2\pi(0)(N-1)}{N}} & e^{-\frac{j2\pi(1)(N-1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix}$$

The conjugate transpose form of matrix A is

$$A^* = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{\frac{j2\pi(0)(0)}{N}} & e^{\frac{j2\pi(0)(1)}{N}} & \dots & e^{\frac{j2\pi(0)(N-1)}{N}} \\ e^{\frac{j2\pi(1)(0)}{N}} & e^{\frac{j2\pi(1)(1)}{N}} & \dots & e^{\frac{j2\pi(1)(N-1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{j2\pi(N-1)(0)}{N}} & e^{\frac{j2\pi(N-1)(1)}{N}} & \dots & e^{\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix}$$

Definition-3

Suppose V is a complex Hilbert space. A bounded linear operator $T: V \rightarrow V$ is said to be unitary if

$$T^*T = TT^* = I$$

With T^* the conjugate transpose.

Based on definition-3, obtained

$$A^*A = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{\frac{j2\pi(0)(0)}{N}} & e^{\frac{j2\pi(0)(1)}{N}} & \dots & e^{\frac{j2\pi(0)(N-1)}{N}} \\ e^{\frac{j2\pi(1)(0)}{N}} & e^{\frac{j2\pi(1)(1)}{N}} & \dots & e^{\frac{j2\pi(1)(N-1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{j2\pi(N-1)(0)}{N}} & e^{\frac{j2\pi(N-1)(1)}{N}} & \dots & e^{\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{N}} \begin{bmatrix} e^{-\frac{j2\pi(0)(0)}{N}} & e^{-\frac{j2\pi(1)(0)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(0)}{N}} \\ e^{-\frac{j2\pi(0)(1)}{N}} & e^{-\frac{j2\pi(1)(1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{j2\pi(0)(N-1)}{N}} & e^{-\frac{j2\pi(1)(N-1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix}$$

$$= \frac{1}{N} \begin{bmatrix} \sum_{s=0}^{N-1} e^{\frac{j2\pi(0)(s)-(s)(0)}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi(0)(s)-(1)(s)}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi(0)(s)-(N-1)(s)}{N}} \\ \sum_{s=0}^{N-1} e^{\frac{j2\pi(1)(s)-(s)(0)}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi(1)(s)-(1)(s)}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi(1)(s)-(N-1)(s)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(s)(0)}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(1)(s)}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(N-1)(s)}{N}} \end{bmatrix}$$

$$= \frac{1}{N} \begin{bmatrix} \sum_{s=0}^{N-1} e^{\frac{j2\pi(0)}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi(-(s)}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi(-(N-1)(s)}{N}} \\ \sum_{s=0}^{N-1} e^{\frac{j2\pi(s)}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi((1)(s)-(N-1)(s)}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(N-1)(s)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(N-1)(s)}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi(-(N-1)(s)}{N}} \end{bmatrix} \quad (4)$$

(4.1)

Note that for $e^0 = 1$, so that $\sum_{s=0}^{N-1} e^0 = N$, and for $e^{\frac{j2\pi p}{N}}$ with

$$\sum_{s=0}^{N-1} e^{j2\pi s} = e^0 + e^{\frac{j2\pi 1}{N}} + e^{\frac{j2\pi 2}{N}} + \dots + e^{\frac{j2\pi (N-1)}{N}}$$

$$\sum_{s=0}^{N-1} e^{j2\pi s} = \frac{e^0 \left(\left(e^{\frac{j2\pi 1}{N}} \right)^N - 1 \right)}{e^{\frac{j2\pi 1}{N}} - 1} = \frac{1(1-1)}{e^{\frac{j2\pi 1}{N}} - 1} = 0$$

So that

$$A^*A = \frac{1}{N} \begin{bmatrix} N & 0 & \dots & 0 \\ 0 & N & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} = I$$

The other side

$$\begin{aligned} AA^* &= \frac{1}{\sqrt{N}} \begin{bmatrix} e^{-\frac{j2\pi(0)(0)}{N}} & e^{-\frac{j2\pi(1)(0)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(0)}{N}} \\ e^{-\frac{j2\pi(0)(1)}{N}} & e^{-\frac{j2\pi(1)(1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{j2\pi(0)(N-1)}{N}} & e^{-\frac{j2\pi(1)(N-1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix} \\ AA^* &= \frac{1}{\sqrt{N}} \begin{bmatrix} e^{\frac{j2\pi(0)(0)}{N}} & e^{\frac{j2\pi(0)(1)}{N}} & \dots & e^{\frac{j2\pi(0)(N-1)}{N}} \\ e^{\frac{j2\pi(1)(0)}{N}} & e^{\frac{j2\pi(1)(1)}{N}} & \dots & e^{\frac{j2\pi(1)(N-1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{j2\pi(N-1)(0)}{N}} & e^{\frac{j2\pi(N-1)(1)}{N}} & \dots & e^{\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix} \\ &= \frac{1}{N} \begin{bmatrix} \sum_{s=0}^{N-1} e^{\frac{j2\pi((0))}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi(-(s))}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi(-(N-1)(s))}{N}} \\ \sum_{s=0}^{N-1} e^{\frac{j2\pi(s)}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi((1)(s)-(N-1)(s))}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(N-1)(s))}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s))}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(N-1)(s))}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(N-1)(s))}{N}} \end{bmatrix} \\ AA^* &= \frac{1}{N} \begin{bmatrix} N & 0 & \dots & 0 \\ 0 & N & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} = I \end{aligned}$$

Since it has been proven that $AA^* = A^*A = I$. So that $\mathcal{F}$ is a unitary operator. Since $\mathcal{F}_A$ is a unitary operator with transformation matrix A , then

$$A^* = A^{-1}$$

Therefore, the DFT with operator $\mathcal{F}$ has an inverse; then, after knowing the existence of the DFT inverse, the next step is to prove proposition-1.

Proposition-1

Inverse form of DFT

$$\forall n = 0, \dots, N-1, \quad f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{\frac{j2\pi nk}{N}}$$

We define a DFT of a sample

$$f \stackrel{\text{def}}{=} \{f_n\}_{n=0}^{N-1}$$

(possibly complex) as a vector of

$$F = \{F_k\}_{k=0}^{N-1} \in \mathbb{C}^N$$

with

$$F_k \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}}$$

To find the inverse form of the DFT. The structure of the DFT inverse mapping is in the form.

$$\mathcal{F}^{-1}: \begin{cases} \mathbb{C}^N \rightarrow \mathbb{C}^N \\ F \mapsto \mathcal{F}(F) = f \end{cases}$$

Therefore, the inverse DFT in the transformation matrix representation will yield.

$$\mathcal{F}^{-1}(F) = A^{-1}F \quad (4.2)$$

Then, the inverse form of DFT can be obtained as follows

$$A^*A = \begin{bmatrix} \sum_{s=0}^{N-1} e^{\frac{j2\pi((0))}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi(-(s))}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi(-(N-1)(s))}{N}} \\ \sum_{s=0}^{N-1} e^{\frac{j2\pi(s)}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi((1)(s)-(N-1)(s))}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(N-1)(s))}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s))}{N}} & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(N-1)(s))}{N}} & \dots & \sum_{s=0}^{N-1} e^{\frac{j2\pi((N-1)(s)-(N-1)(s))}{N}} \end{bmatrix}$$

$$A^*A = \begin{bmatrix} N & 0 & \dots & 0 \\ 0 & N & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & N \end{bmatrix} = NI$$

$$A^*A = NI$$

$$A^* = NA^{-1}$$

$$A^{-1} = \frac{1}{N} A^* \quad (4.3)$$

Because

$$A = \begin{bmatrix} e^{-\frac{j2\pi(0)(0)}{N}} & e^{-\frac{j2\pi(1)(0)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(0)}{N}} \\ e^{-\frac{j2\pi(0)(1)}{N}} & e^{-\frac{j2\pi(1)(1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\frac{j2\pi(0)(N-1)}{N}} & e^{-\frac{j2\pi(1)(N-1)}{N}} & \dots & e^{-\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix}$$

then

$$A^* = \begin{bmatrix} e^{\frac{j2\pi(0)(0)}{N}} & e^{\frac{j2\pi(0)(1)}{N}} & \dots & e^{\frac{j2\pi(0)(N-1)}{N}} \\ e^{\frac{j2\pi(1)(0)}{N}} & e^{\frac{j2\pi(1)(1)}{N}} & \dots & e^{\frac{j2\pi(1)(N-1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{j2\pi(N-1)(0)}{N}} & e^{\frac{j2\pi(N-1)(1)}{N}} & \dots & e^{\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix}$$

So, based on equation (4.3)

$$A^{-1} = \frac{1}{N} \begin{bmatrix} e^{\frac{j2\pi(0)(0)}{N}} & e^{\frac{j2\pi(0)(1)}{N}} & \dots & e^{\frac{j2\pi(0)(N-1)}{N}} \\ e^{\frac{j2\pi(1)(0)}{N}} & e^{\frac{j2\pi(1)(1)}{N}} & \dots & e^{\frac{j2\pi(1)(N-1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{j2\pi(N-1)(0)}{N}} & e^{\frac{j2\pi(N-1)(1)}{N}} & \dots & e^{\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix}$$

Therefore, by using equation (4.2)

$$\begin{aligned} \mathcal{F}^{-1}(F) &= \frac{1}{N} \begin{bmatrix} e^{\frac{j2\pi(0)(0)}{N}} & e^{\frac{j2\pi(0)(1)}{N}} & \dots & e^{\frac{j2\pi(0)(N-1)}{N}} \\ e^{\frac{j2\pi(1)(0)}{N}} & e^{\frac{j2\pi(1)(1)}{N}} & \dots & e^{\frac{j2\pi(1)(N-1)}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{j2\pi(N-1)(0)}{N}} & e^{\frac{j2\pi(N-1)(1)}{N}} & \dots & e^{\frac{j2\pi(N-1)(N-1)}{N}} \end{bmatrix} \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{N-1} \end{bmatrix} \\ \mathcal{F}^{-1}(F) &= \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{\frac{j2\pi nk}{N}} \end{aligned}$$

So, the inverse DFT is in the form

$$\mathcal{F}^{-1}(F) = f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n e^{\frac{j2\pi nk}{N}}.$$

C. DFT Properties

a. Linearity

The linearity property is self-evident since the DFT is a linear operator, but perhaps what has not been proven here is its inverse linear form. For example,

$$\mathbb{C}^N = \{\mathbf{F}^{(m)} = (F_0, F_1, \dots, F_k^{(m)}, \dots, F_{N-1}) | F_k \in \mathbb{C}, \forall k = 0, 1, \dots, N-1, m \in \mathbb{N}\}$$

and form a function so that

$$\mathbf{F}^{(3)} = c_1 \mathbf{F}^{(1)} + c_2 \mathbf{F}^{(2)} \in \mathbb{C}^N$$

with c_1, c_2 scalar, such that

$$\mathcal{F}^{-1}(\mathbf{F}^{(3)}) = \mathbf{f}_n = \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{F}^{(3)} e^{\frac{j2\pi nk}{N}} \quad (0 \leq n \leq N-1)$$

$$\mathcal{F}^{-1}(c_1 \mathbf{F}^{(1)} + c_2 \mathbf{F}^{(2)}) = \frac{1}{N} \sum_{k=0}^{N-1} (c_1 F_k^{(1)} + c_2 F_k^{(2)}) e^{\frac{j2\pi nk}{N}}$$

$$\mathcal{F}^{-1}(c_1 F_k^{(1)} + c_2 F_k^{(2)}) = \frac{1}{N} \sum_{k=0}^{N-1} c_1 F_k^{(1)} e^{\frac{j2\pi nk}{N}} + c_2 F_k^{(2)} e^{\frac{j2\pi nk}{N}}$$

$$\begin{aligned} \mathcal{F}^{-1}(c_1 F_k^{(1)} + c_2 F_k^{(2)}) &= \frac{1}{N} \sum_{k=0}^{N-1} c_1 F_k^{(1)} e^{\frac{j2\pi nk}{N}} \\ &+ \frac{1}{N} \sum_{k=0}^{N-1} c_2 F_k^{(2)} e^{\frac{j2\pi nk}{N}} \end{aligned}$$

$$\begin{aligned} \mathcal{F}^{-1}(c_1 F_k^{(1)} + c_2 F_k^{(2)}) &= \frac{c_1}{N} \sum_{k=0}^{N-1} F_k^{(1)} e^{\frac{j2\pi nk}{N}} + \frac{c_2}{N} \sum_{k=0}^{N-1} F_k^{(2)} e^{\frac{j2\pi nk}{N}} \\ \mathcal{F}^{-1}(c_1 F_k^{(1)} + c_2 F_k^{(2)}) &= c_1 f_k^{(1)} + c_2 f_k^{(2)} \end{aligned}$$

b. Periodicity. It will be proven that

$$\begin{aligned} F_{n+N} &= F_n \\ F_n &= \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi nk}{N}} \\ F_{n+N} &= \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi(n+N)k}{N}} = \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi nk - j2\pi Nk}{N}} \\ F_{n+N} &= \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi nk}{N}} \cdot e^{\frac{-j2\pi Nk}{N}} \\ F_{n+N} &= \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi nk}{N}} e^{-j2\pi k} \\ F_{n+N} &= \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi nk}{N}} = F_n \end{aligned}$$

c. Symmetry

(a) Suppose f_k is real for every $0 \leq k \leq N-1$

$$\begin{aligned} F_n &= \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi nk}{N}} \\ F_n &= \sum_{k=0}^{N-1} f_k \left(\cos\left(\frac{-2\pi nk}{N}\right) + j \sin\left(\frac{-2\pi nk}{N}\right) \right) \end{aligned}$$

The real and imaginary parts of F_n are presented in the following equations

$$\text{Re}(F_n) = \sum_{k=0}^{N-1} f_k \cos\left(\frac{-2\pi nk}{N}\right) \quad (4.4)$$

$$\text{Im}(F_n) = \sum_{k=0}^{N-1} f_k \sin\left(\frac{-2\pi nk}{N}\right) \quad (4.5)$$

Reform equations (4.4) and (4.5) above into functions

$$g(n) = \text{Re}(F_n) = \sum_{k=0}^{N-1} f_k \cos\left(\frac{-2\pi nk}{N}\right)$$

$$g(n) = \sum_{k=0}^{N-1} f_k \cos\left(\frac{2\pi nk}{N}\right)$$

$$h(n) = \text{Im}(F_n) = \sum_{k=0}^{N-1} f_k \sin\left(\frac{-2\pi nk}{N}\right)$$

$$h(n) = - \sum_{k=0}^{N-1} f_k \sin\left(\frac{2\pi nk}{N}\right)$$

note that

$$g(-n) = \sum_{k=0}^{N-1} f_k \cos\left(\frac{2\pi(-n)k}{N}\right)$$

$$g(-n) = \sum_{k=0}^{N-1} f_k \cos\left(\frac{2\pi nk}{N}\right) = g(n)$$

$$h(-n) = - \sum_{k=0}^{N-1} f_k \sin\left(\frac{2\pi(-n)k}{N}\right)$$

$$h(-n) = \sum_{k=0}^{N-1} f_k \sin\left(\frac{2\pi nk}{N}\right) = -h(n)$$

So, for real f_k we get $\text{Re}(F_n)$ an even function and $\text{Im}(F_n)$ an odd function.

(b) For example, f_k of imaginary was the component of $\mathbf{f}$ with $f_k = ja_k$

$$\begin{aligned} F_n &= \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi nk}{N}} \\ F_n &= \sum_{k=0}^{N-1} ja_k \left(\cos\left(\frac{-2\pi nk}{N}\right) + j \sin\left(\frac{-2\pi nk}{N}\right) \right) \\ F_n &= \sum_{k=0}^{N-1} \left(ja_k \cos\left(\frac{-2\pi nk}{N}\right) - a_k \sin\left(\frac{-2\pi nk}{N}\right) \right) \quad (4.6) \end{aligned}$$

Based on equation (4.6)

$$g(n) = \text{Re}(F_n) = \sum_{k=0}^{N-1} \left(-a_k \sin\left(\frac{-2\pi nk}{N}\right) \right)$$

$$g(n) = \sum_{k=0}^{N-1} \left(a_k \sin\left(\frac{2\pi nk}{N}\right) \right)$$

$$h(n) = \text{Im}(F_n) = \sum_{k=0}^{N-1} \left(a_k \cos\left(\frac{-2\pi nk}{N}\right) \right)$$

$$h(n) = \sum_{k=0}^{N-1} \left(a_k \cos\left(\frac{2\pi nk}{N}\right) \right)$$

the other side

$$\begin{aligned} g(-n) &= \sum_{k=0}^{N-1} \left(a_k \sin\left(\frac{2\pi(-n)k}{N}\right) \right) \\ g(-n) &= - \sum_{k=0}^{N-1} \left(a_k \sin\left(\frac{2\pi nk}{N}\right) \right) = -g(n) \\ h(-n) &= \sum_{k=0}^{N-1} \left(a_k \cos\left(\frac{2\pi(-n)k}{N}\right) \right) \\ h(-n) &= \sum_{k=0}^{N-1} \left(a_k \cos\left(\frac{2\pi nk}{N}\right) \right) = h(n) \end{aligned}$$

so that $Re(F_n)$ is an odd function and $Im(F_n)$ is an even function if f_k is imaginary

(c) If f_k is real and the function is even then

$$\begin{aligned} f_{-k} &= f_{-k+N} = f_k \\ F_n &= \sum_{k=0}^{N-1} f_k e^{-j\frac{2\pi nk}{N}} \\ F_n &= \sum_{k=0}^{N-1} f_k \left(\cos\left(\frac{-2\pi nk}{N}\right) + j \sin\left(\frac{-2\pi nk}{N}\right) \right) \\ F_n &= \sum_{k=0}^{N-1} f_k \left(\cos\left(\frac{2\pi nk}{N}\right) - j \sin\left(\frac{2\pi nk}{N}\right) \right) \end{aligned}$$

The other side

$$\begin{aligned} F_n &= \sum_{k=0}^{N-1} f_{-k} e^{-j\frac{2\pi n(-k)}{N}} \\ F_n &= \sum_{k=0}^{N-1} f_k \left(\cos\left(\frac{2\pi nk}{N}\right) + j \sin\left(\frac{2\pi nk}{N}\right) \right) \\ \text{Add up the two } F_n \text{ to get} \\ 2F_n &= \sum_{k=0}^{N-1} f_k \left(2 \cos\left(\frac{2\pi nk}{N}\right) \right) \\ F_n &= \sum_{k=0}^{N-1} f_k \left(\cos\left(\frac{2\pi nk}{N}\right) \right) = \sum_{k=0}^{N-1} f_k \left(\cos\left(\frac{2\pi nk}{N}\right) \right) \\ F_{-n} &= \sum_{k=0}^{N-1} f_k \left(\cos\left(\frac{2\pi(-n)k}{N}\right) \right) = F_n \end{aligned}$$

So that F is real and even

(d) If f_k is real and odd, then $f_k = -f_{-k+N} = -f_{-k}$

$$\begin{aligned} F_n &= \sum_{k=0}^{N-1} f_k e^{-j\frac{2\pi nk}{N}} \\ F_n &= \sum_{k=0}^{N-1} f_k \left(\cos\left(\frac{2\pi nk}{N}\right) - j \sin\left(\frac{2\pi nk}{N}\right) \right) \end{aligned}$$

By replacing k with $-k$

$$\begin{aligned} F_n &= \sum_{k=0}^{N-1} f_{-k} \left(\cos\left(\frac{2\pi n(-k)}{N}\right) - j \sin\left(\frac{2\pi n(-k)}{N}\right) \right) \\ F_n &= \sum_{k=0}^{N-1} f_{-k} \left(\cos\left(\frac{2\pi nk}{N}\right) + j \sin\left(\frac{2\pi nk}{N}\right) \right) \\ \text{Add up the values of } F_n, \text{ so that} \\ 2F_n &= \sum_{k=0}^{N-1} f_{-k} \left(\cos\left(\frac{2\pi nk}{N}\right) + j \sin\left(\frac{2\pi nk}{N}\right) \right) \\ &\quad + \sum_{k=0}^{N-1} f_k \left(\cos\left(\frac{2\pi nk}{N}\right) - j \sin\left(\frac{2\pi nk}{N}\right) \right) \end{aligned}$$

$$\begin{aligned} 2F_n &= \sum_{k=0}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) (f_{-k} + f_k) \\ &\quad + j \sin\left(\frac{2\pi nk}{N}\right) (f_{-k} - f_k) \end{aligned}$$

Since f_k is odd, then

$$f_k = -f_{-k}$$

So that

$$\begin{aligned} 2F_n &= -2 \sum_{k=0}^{N-1} j f_k \sin\left(\frac{2\pi nk}{N}\right) \\ F_n &= - \sum_{k=0}^{N-1} j f_k \sin\left(\frac{2\pi nk}{N}\right) \end{aligned}$$

So F_n is imaginary. Then, replace n with $-n$ such that

$$\begin{aligned} F_{-n} &= - \sum_{k=0}^{N-1} j f_k \sin\left(\frac{-2\pi nk}{N}\right) \\ F_{-n} &= \sum_{k=0}^{N-1} j f_k \sin\left(\frac{2\pi nk}{N}\right) = -F_n \end{aligned}$$

So F_n is odd and imaginary.

(e) If f_k is imaginary with $f_k = j a_k$ and even, that is,

$$\begin{aligned} f_k &= f_{-k+N} = f_{-k} \\ F_n &= \sum_{k=0}^{N-1} f_k e^{-j\frac{2\pi nk}{N}} \\ F_n &= \sum_{k=0}^{N-1} j a_k \left(\cos\left(\frac{-2\pi nk}{N}\right) + j \sin\left(\frac{-2\pi nk}{N}\right) \right) \\ F_n &= \sum_{k=0}^{N-1} \left(j a_k \cos\left(\frac{2\pi nk}{N}\right) + a_k \sin\left(\frac{2\pi nk}{N}\right) \right) \end{aligned}$$

By replacing k with $-k$

$$\begin{aligned}
F_n &= \sum_{k=0}^{N-1} \left(ja_{-k} \cos\left(\frac{2\pi n(-k)}{N}\right) \right. \\
&\quad \left. + a_{-k} \sin\left(\frac{2\pi n(-k)}{N}\right) \right) \\
F_n &= \sum_{k=0}^{N-1} \left(ja_{-k} \cos\left(\frac{2\pi nk}{N}\right) - a_{-k} \sin\left(\frac{2\pi nk}{N}\right) \right) \\
\text{Sum the values of } F_n \\
2F_n &= \sum_{k=0}^{N-1} \left(ja_k \cos\left(\frac{2\pi nk}{N}\right) + a_k \sin\left(\frac{2\pi nk}{N}\right) \right) \\
&\quad + \sum_{k=0}^{N-1} \left(ja_{-k} \cos\left(\frac{2\pi nk}{N}\right) \right. \\
&\quad \left. - a_{-k} \sin\left(\frac{2\pi nk}{N}\right) \right) \\
2F_n &= \sum_{k=0}^{N-1} \left(j \cos\left(\frac{2\pi nk}{N}\right) (a_k + a_{-k}) \right. \\
&\quad \left. + \sin\left(\frac{2\pi nk}{N}\right) (a_k - a_{-k}) \right)
\end{aligned}$$

Because f_k is an even function, then $f_k = f_{-k}$
 As a result, $a_k = a_{-k}$

$$\begin{aligned}
2F_n &= \sum_{k=0}^{N-1} \left(j2a_k \cos\left(\frac{2\pi nk}{N}\right) \right) \\
F_n &= \sum_{k=0}^{N-1} \left(ja_k \cos\left(\frac{2\pi nk}{N}\right) \right) \\
F_{-n} &= \sum_{k=0}^{N-1} \left(ja_k \cos\left(\frac{-2\pi nk}{N}\right) \right) \\
F_{-n} &= \sum_{k=0}^{N-1} \left(ja_k \cos\left(\frac{2\pi nk}{N}\right) \right) = F_n
\end{aligned}$$

So F_n is imaginary and even.

- (f) If f_k is imaginary and is an odd function, then $f_k = -f_{-k+N} = -f_{-k}$

$$\begin{aligned}
F_n &= \sum_{k=0}^{N-1} f_k e^{\frac{-j2\pi nk}{N}} \\
F_n &= \sum_{k=0}^{N-1} ja_k \left(\cos\left(\frac{-2\pi nk}{N}\right) + j \sin\left(\frac{-2\pi nk}{N}\right) \right) \\
F_n &= \sum_{k=0}^{N-1} \left(ja_k \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \\
\text{By replacing } k \text{ with } -k \\
F_n &= \sum_{k=0}^{N-1} \left(ja_{-k} \cos\left(\frac{2\pi n(-k)}{N}\right) \right. \\
&\quad \left. + a_{-k} \sin\left(\frac{2\pi n(-k)}{N}\right) \right)
\end{aligned}$$

$$\begin{aligned}
F_n &= \sum_{k=0}^{N-1} \left(ja_{-k} \cos\left(\frac{2\pi nk}{N}\right) - a_{-k} \sin\left(\frac{2\pi nk}{N}\right) \right) \\
\text{Add up } F_n \\
2F_n &= \sum_{k=0}^{N-1} \left(j \cos\left(\frac{2\pi nk}{N}\right) (a_k + a_{-k}) \right. \\
&\quad \left. + \sin\left(\frac{2\pi nk}{N}\right) (a_k - a_{-k}) \right)
\end{aligned}$$

Since f_k is an odd function, meaning $a_k = -a_{-k}$

$$\begin{aligned}
2F_n &= \sum_{k=0}^{N-1} \left(2a_k \sin\left(\frac{2\pi nk}{N}\right) \right) \\
F_n &= \sum_{k=0}^{N-1} \left(a_k \sin\left(\frac{2\pi nk}{N}\right) \right) \\
F_{-n} &= \sum_{k=0}^{N-1} \left(a_k \sin\left(\frac{-2\pi nk}{N}\right) \right) \\
F_{-n} &= - \sum_{k=0}^{N-1} \left(a_k \sin\left(\frac{2\pi nk}{N}\right) \right) = -F_n
\end{aligned}$$

So that F_n is real function and odd function.

D. Discretization of Continuous Functions

The process of transforming continuous data $f(t)$ into discrete data in this study will utilize the impulse sampling method. This method uses the Dirac delta series, as shown in definition-4.

Definition-4

Dirac delta is a distribution defined as

$$\begin{aligned}
\delta(t) &= \lim_{k \rightarrow 0} B_k(t) \\
\delta_T(t) &= \sum_{n=-\infty}^{\infty} \delta(t - nT_s).
\end{aligned}$$

The sampling technique that will be used is impulse sampling using the Dirac delta. Impulse sampling is a mathematical procedure by multiplying the desired continuous function as a sample by the impulse train $\delta_T(t)$ which is defined as

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Since the Dirac delta cannot be realized geometrically, its representation will be approximated by the pulse box function in Definition-5.

Definition-5

Pulse, with length k and height 1 is called $\text{Rect}\left(\frac{t}{k}\right)$ whit

$$\text{Rect}\left(\frac{t}{k}\right) = \begin{cases} 1, & |t| < \frac{k}{2} \\ \frac{1}{2}, & |t| = \frac{k}{2} \\ 0, & |t| > \frac{k}{2}. \end{cases}$$

The function has a length of k , the representation of the Dirac delta in this study uses the box function which is defined as follows.

$$B_k(t) = \frac{1}{k} \text{Rect}\left(\frac{t}{k}\right)$$

This method is a mathematical procedure that multiplies a continuous function by the Dirac delta series, which means that each sample point selected from the continuous function will be brought by the Dirac delta to the value corresponding to the function, so that the form of the sampling model is in the form of a sum.

$$x(t)\delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Definition-6

Dirac delta or unit impulse is defined

$$\delta(t) = 0, t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

According to Definition-6, The Dirac delta $\delta(t)$ is zero everywhere except at $t = 0$, so $\delta(t - nT_s)$ is zero everywhere except at $t = nT_s$, with n was an integer and T_s the distance between samples. So that the form of the sampling model is

$$x_s(t) = x(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \quad (4.6)$$

For example, if there are N samples, then the sample value model becomes

$$x_{t_1}(t) = x(t_1)\delta(t - t_1)$$

$$x_{t_2}(t) = x(t_2)\delta(t - t_2)$$

...

$$x_{t_N}(t) = x(t_N)\delta(t - t_N)$$

With x_{t_n} , the function value of the t_n sample or the amplitude of the to- n Dirac delta. For example, the function

$$f(t) = \begin{cases} \frac{\sin(t)}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

The graph of the function $f(t)$ is shown in Figure 4

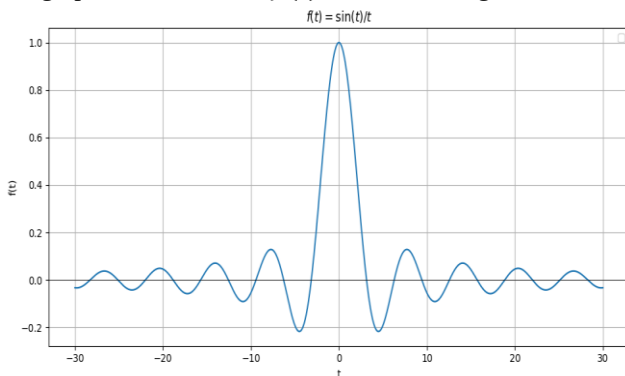


Figure 3. Graph of $f(t) = \frac{\sin(t)}{t}$

For example, $N = 61$ samples will be selected with the distance between samples $T_s = 1$ in the interval $[-30, 30]$ and

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(nT_s)\delta(t - nT_s)$$

it values in the Dirac delta are as many as the chosen samples. Figure 5 below shows the impulse series values.

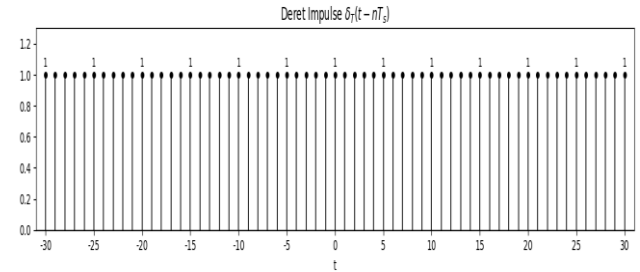


Figure 4. Existence of impulse value

The impulse series will bring the value $f(t)$ from the selected sample point to the value corresponding to $f(t)$, and values that are not selected are considered non-existent or not carried by the Dirac delta. So, the Dirac delta amplitude is calculated from the following equation (4.7).

$$f_s(t) = f(-30)\delta(t + 30) + f(-29)\delta(t + 29) + \dots + f(30)\delta(t - 30)$$

$$f_s(t) = \frac{\sin(-30)}{-30}\delta(t + 30) + \frac{\sin(-29)}{-29}\delta(t + 29) + \dots + \frac{\sin(30)}{30}\delta(t - 30)$$

Some amplitudes of each Dirac delta are shown in Figure 6.

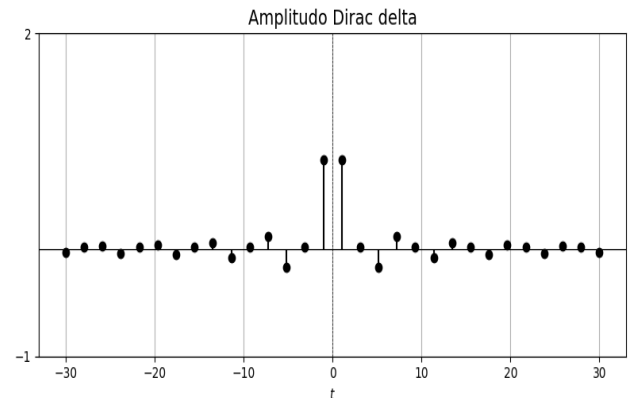
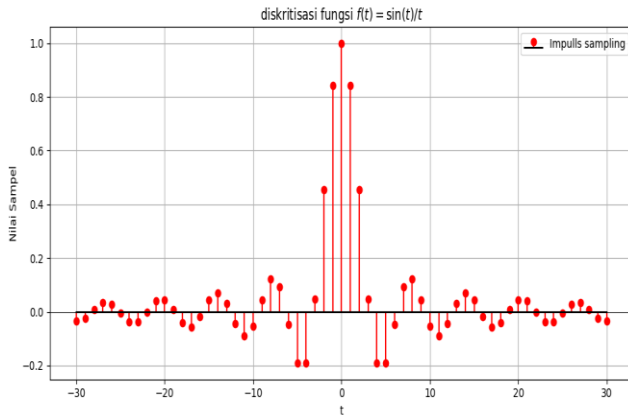


Figure 5. Dirac delta amplitude

The result of discretization the function $f(t)$ is shown in the following Figure 7.

Figure 6. Function sampling results of $f(t)$

E. Application of DFT

One application of DFT in the biomedical field is the processing of EEG signals. The brain waves measured by an EEG are naturally continuous. Therefore, EEG devices typically facilitate the transformation of constant signals into discrete ones. The EEG signal output measures waves in the time domain in discrete form [16], normally programmed into the device with a sampling rule. For example, an EEG device with a sampling frequency of 160 Hz. This means the EEG will record 160 brain activity samples per second. DFT can be used to process discrete-based EEG data in the time domain [17]. DFT processes discrete EEG signals in the time domain to convert them into the frequency domain [18]. Frequency-domain analysis of EEG signals provides valuable insights into neurological and psychological conditions. The purpose of DFT in frequency analysis is to calculate the magnitude (amplitude) and energy, which can provide conclusions about brain conditions.

For example, a mathematical model of brain waves is built using a sinusoidal approach consisting of several frequency components in the range $\nu_i \in [16 - 30]$ Hz, which is the range in the medical and neurophysiological world, and an amplitude of $10 - 100\mu V$ or $0.01 - 0.1mV$ is classified as a beta wave. Beta waves appear when a person is awake, focused, and actively thinking. The mathematical model in this study is constructed from several sine waves assumed to be 'similar' or close to the beta waves in Figure 3, each lasting 1 second under the specified condition. The wave equation model is shown as follows

$$f(t) = \sum_{i=1}^{10} A_i \cdot \sin(2\pi\nu_i t + \phi_i)$$

$$\begin{aligned} f(t) = & 0.07 \sin(2\pi(16)t) + 0.08 \sin(2\pi(17)t) \\ & + 0.05 \sin\left(2\pi(18)t + \frac{\pi}{6}\right) \\ & + 0.04 \sin\left(2\pi(19)t + \frac{\pi}{6}\right) \\ & + 0.03 \sin\left(2\pi(20)t + \frac{\pi}{3}\right) \\ & + 0.09 \sin\left(2\pi(21)t + \frac{\pi}{3}\right) \\ & + 0.012 \sin\left(2\pi(22)t + \frac{\pi}{2}\right) \\ & + 0.02 \sin\left(2\pi(26)t + \frac{\pi}{2}\right) \\ & + 0.099 \sin\left(2\pi(28)t + \frac{\pi}{4}\right) \\ & + 0.01 \sin\left(2\pi(30)t + \frac{\pi}{4}\right) \end{aligned} \quad (4.7)$$

$$\begin{aligned} f(t) = & 0.07 \sin(32\pi t) + 0.08 \sin(34\pi t) \\ & + 0.05 \sin\left(36\pi t + \frac{\pi}{6}\right) \\ & + 0.04 \sin\left(38\pi t + \frac{\pi}{6}\right) \\ & + 0.03 \sin\left(40\pi t + \frac{\pi}{3}\right) \\ & + 0.09 \sin\left(42\pi t + \frac{\pi}{3}\right) \\ & + 0.012 \sin\left(44\pi t + \frac{\pi}{2}\right) \\ & + 0.02 \sin\left(52\pi t + \frac{\pi}{2}\right) \\ & + 0.099 \sin\left(56\pi t + \frac{\pi}{4}\right) \\ & + 0.01 \sin\left(60\pi t + \frac{\pi}{4}\right) \end{aligned} \quad (4.8)$$

The continuous model graph of equation (4.8) is shown in the following Figure 7.

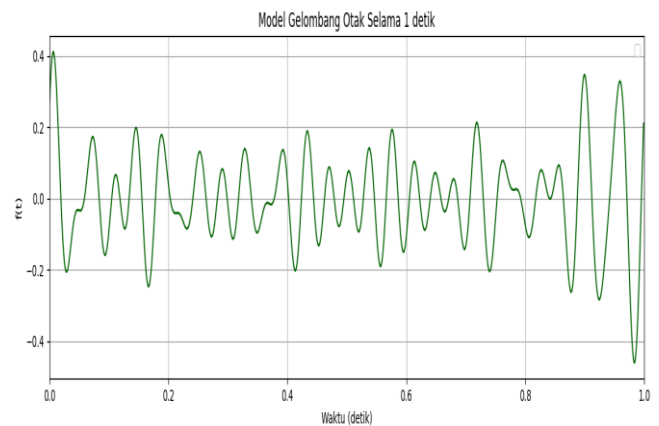
Figure 7. Graph of the function $f(t)$

Figure 7. Graph of the function $f(t)$ from equation (4.8). The equation model (4.8) will meet the Nyquist criterion if $\nu_s > 2\nu_{max}$. From the model, it can be seen that the largest frequency is at $\nu_{max} = 30$ Hz; therefore, the sampling frequency of the model must $\nu_s > 60$ be greater than 60 Hz. This is important to avoid aliasing and to take the distance between samples. Next, the model in equation (4.8) will be discretized using impulse sampling in the interval $[0,1]$. For example, the sampling frequency $\nu_s = 1000$ in this case

means that 1000 samples are taken per second $T_s = \frac{1}{1000}$ seconds, or 1 millisecond, a sample is taken. Thus, the sampling points follow the form $t_s = 0 + \frac{1}{1000}s = \frac{s}{1000}$, where $s = 0, 1, 2, \dots, N-1$. nT_s represents the selected sample, so in this case, $t_s = n$, $T_s = \frac{s}{1000}$, consequently $n = s$. Therefore, based on equation (4.7) adjusted to a nonnegative n value, the sampling model is

$$f_s(t) = \sum_{n=0}^{N-1} \left[0.07 \sin(32\pi t) + 0.08 \sin(34\pi t) + 0.05 \sin\left(36\pi t + \frac{\pi}{6}\right) + 0.04 \sin\left(38\pi t + \frac{\pi}{6}\right) + 0.03 \sin\left(40\pi t + \frac{\pi}{3}\right) + 0.09 \sin\left(42\pi t + \frac{\pi}{3}\right) + 0.012 \sin\left(44\pi t + \frac{\pi}{2}\right) + 0.02 \sin\left(52\pi t + \frac{\pi}{2}\right) + 0.099 \sin\left(56\pi t + \frac{\pi}{4}\right) + 0.01 \sin\left(60\pi t + \frac{\pi}{4}\right) \right] \delta(t - nT_s) \quad (4.8)$$

Because the Dirac delta only carries the selected sample, while outside the sample the Dirac delta will bring its value to zero, then

$$f_s(nT_s) = \sum_{n=0}^{N-1} \left[0.07 \sin(32\pi nT_s) + 0.08 \sin(34\pi nT_s) + 0.05 \sin\left(36\pi nT_s + \frac{\pi}{6}\right) + 0.04 \sin\left(38\pi nT_s + \frac{\pi}{6}\right) + 0.03 \sin\left(40\pi nT_s + \frac{\pi}{3}\right) + 0.09 \sin\left(42\pi nT_s + \frac{\pi}{3}\right) + 0.012 \sin\left(44\pi nT_s + \frac{\pi}{2}\right) + 0.02 \sin\left(52\pi nT_s + \frac{\pi}{2}\right) + 0.099 \sin\left(56\pi nT_s + \frac{\pi}{4}\right) + 0.01 \sin\left(60\pi nT_s + \frac{\pi}{4}\right) \right] \delta(t - nT_s) \quad (4.9)$$

because $t \in [0, 1]$ and the sample selection follows the form $nT_s = \frac{s}{1000}$, namely $nT_s \left\{ \frac{0}{1000}, \frac{1}{1000}, \frac{2}{1000}, \dots, \frac{N-1}{1000} \right\}$, the number of samples is

$$\begin{aligned} \frac{N-1}{1000} &= 1 \\ N-1 &= 1000 \\ N &= 1001 \end{aligned}$$

So that the form of equation (4.10) can be simplified to

$$f_s = \sum_{n=0}^{1000} \left[0.07 \sin(32\pi nT_s) + 0.08 \sin(34\pi nT_s) + 0.05 \sin\left(36\pi nT_s + \frac{\pi}{6}\right) + 0.04 \sin\left(38\pi nT_s + \frac{\pi}{6}\right) + 0.03 \sin\left(40\pi nT_s + \frac{\pi}{3}\right) + 0.09 \sin\left(42\pi nT_s + \frac{\pi}{3}\right) + 0.012 \sin\left(44\pi nT_s + \frac{\pi}{2}\right) + 0.02 \sin\left(52\pi nT_s + \frac{\pi}{2}\right) + 0.099 \sin\left(56\pi nT_s + \frac{\pi}{4}\right) + 0.01 \sin\left(60\pi nT_s + \frac{\pi}{4}\right) \right] \delta(t - nT_s) \quad (4.10)$$

With f_s indicating the discrete value or discrete signal to s . The Dirac delta value in equation (4.11) will only be zero everywhere except in the selected sample. For example, the sample chosen is $\frac{1000}{1000}$, indicating that the point is located in the sample at $s = 1000$. Then, the Dirac delta value will be equal to zero everywhere except at $\frac{1000}{1000} = 1$, so that

$$f_{1000} = \sum_{n=0}^{1000} \left[0.07 \sin(32\pi nT_s) + 0.08 \sin(34\pi nT_s) + 0.05 \sin\left(36\pi nT_s + \frac{\pi}{6}\right) + 0.04 \sin\left(38\pi nT_s + \frac{\pi}{6}\right) + 0.03 \sin\left(40\pi nT_s + \frac{\pi}{3}\right) + 0.09 \sin\left(42\pi nT_s + \frac{\pi}{3}\right) + 0.012 \sin\left(44\pi nT_s + \frac{\pi}{2}\right) + 0.02 \sin\left(52\pi nT_s + \frac{\pi}{2}\right) + 0.099 \sin\left(56\pi nT_s + \frac{\pi}{4}\right) + 0.01 \sin\left(60\pi nT_s + \frac{\pi}{4}\right) \right] \delta(t - nT_s)$$

$$\begin{aligned}
 f_{1000} = \sum_{n=0}^{1000} & \left[0.07 \sin\left(\frac{32\pi n}{1000}\right) + 0.08 \sin\left(\frac{34\pi n}{1000}\right) \right. \\
 & + 0.05 \sin\left(\frac{36\pi n}{1000} + \frac{\pi}{6}\right) \\
 & + 0.04 \sin\left(\frac{38\pi n}{1000} + \frac{\pi}{6}\right) \\
 & + 0.03 \sin\left(\frac{40\pi n}{1000} + \frac{\pi}{3}\right) \\
 & + 0.09 \sin\left(\frac{42\pi n}{1000} + \frac{\pi}{3}\right) \\
 & + 0.012 \sin\left(\frac{44\pi n}{1000} + \frac{\pi}{2}\right) \\
 & + 0.02 \sin\left(\frac{52\pi n}{1000} + \frac{\pi}{2}\right) \\
 & + 0.099 \sin\left(\frac{56\pi n}{1000} + \frac{\pi}{4}\right) \\
 & \left. + 0.01 \sin\left(\frac{60\pi n}{1000} + \frac{\pi}{4}\right) \right] \delta\left(t - \frac{n}{1000}\right) \\
 f_{1000} = & \left[0.07 \sin(0) + 0.08 \sin(0) + 0.05 \sin\left(\frac{\pi}{6}\right) \right. \\
 & + 0.04 \sin\left(\frac{\pi}{6}\right) + 0.03 \sin\left(\frac{\pi}{3}\right) \\
 & + 0.09 \sin\left(\frac{\pi}{3}\right) + 0.012 \sin\left(\frac{\pi}{2}\right) \\
 & + 0.02 \sin\left(\frac{\pi}{2}\right) + 0.099 \sin\left(\frac{\pi}{4}\right) \\
 & \left. + 0.01 \sin\left(\frac{\pi}{4}\right) \right] \delta(t - 1) \\
 f_{1000} \approx & 0.259
 \end{aligned}$$

So that the value at the 1000th sample, namely $t = 1$, will be brought to 0.259 by the Dirac delta, which in this case represents the Dirac delta weight at point $t = 1$.

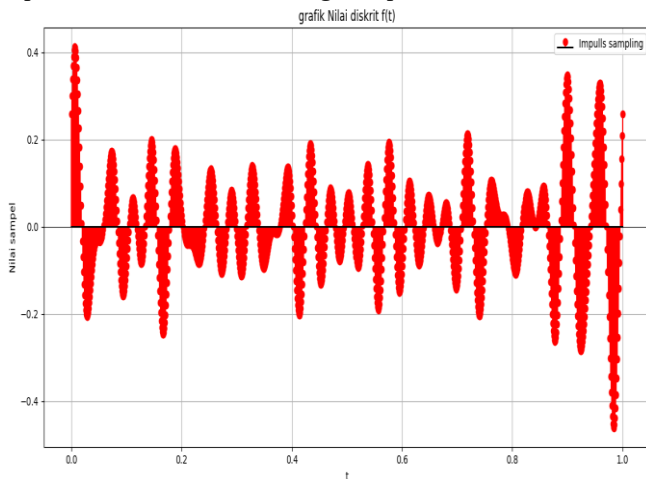


Figure 8. Function graph of f_s

Next, after obtaining a discrete signal, signal processing can be performed in the frequency domain using the DFT, as shown in equation (2.3).

$$F_k \stackrel{\text{def}}{=} \sum_{n=0}^{N-1} f_n e^{-\frac{j2\pi nk}{N}}$$

For example, $f_s = f_n$ because it was previously explained that the index $s = n = 0, 1, \dots, 1000$

$$\begin{aligned}
 F_k &= \sum_{n=0}^{1001-1} f_n e^{-\frac{j2\pi nk}{1001}} \\
 F_k &= \sum_{n=0}^{1000} f_n e^{-\frac{j2\pi nk}{1001}} \\
 F_k &= f_0 e^{-\frac{j2\pi 0k}{1001}} + f_1 e^{-\frac{j2\pi 1k}{1001}} + \dots + f_{1000} e^{-\frac{j2\pi (1000)k}{1001}}
 \end{aligned}$$

Representation of F_k in a matrix form

$$\begin{bmatrix} F_0 \\ F_1 \\ \dots \\ F_{1000} \end{bmatrix} = \begin{bmatrix} e^{-\frac{j2\pi(0)(0)}{1001}} & e^{-\frac{j2\pi(1)(0)}{1001}} & \dots & e^{-\frac{j2\pi(1000)(0)}{1001}} \\ e^{-\frac{j2\pi(0)(1)}{1001}} & e^{-\frac{j2\pi(1)(1)}{1001}} & \dots & e^{-\frac{j2\pi(1000)(1)}{1001}} \\ \dots & \dots & \dots & \dots \\ e^{-\frac{j2\pi(0)(1000)}{1001}} & e^{-\frac{j2\pi(1)(1000)}{1001}} & \dots & e^{-\frac{j2\pi(1000)(1000)}{1001}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \dots \\ f_{1000} \end{bmatrix}$$

The last step is to calculate the magnitude and energy contained in equations (2.8) and (2.9), which are in the form

$$|F_k| = \sqrt{\text{Re}(F_k)^2 + \text{Im}(F_k)^2}$$

$$P_k = |F_k|^2 = \text{Re}(F_k)^2 + \text{Im}(F_k)^2$$

The magnitude and energy graph is shown in the following Figure 10.

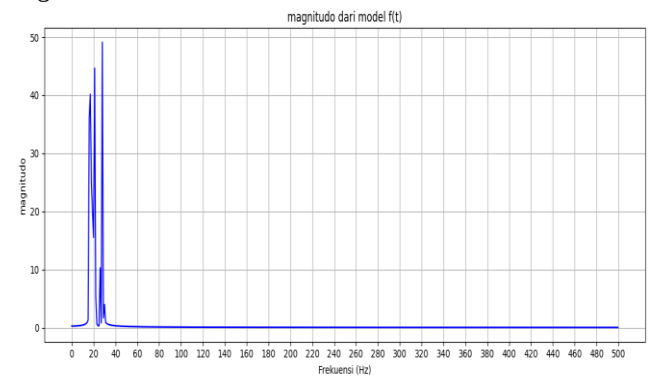


Figure 9. DFT Magnitude Graph

In Figure 9, since the discrete signal is real-valued, according to the previous explanation of the DFT properties, the graph will be symmetrical. Magnitude values can be found in the appendix. When enlarged, the magnitude graph will appear as shown in the following Figure 10.

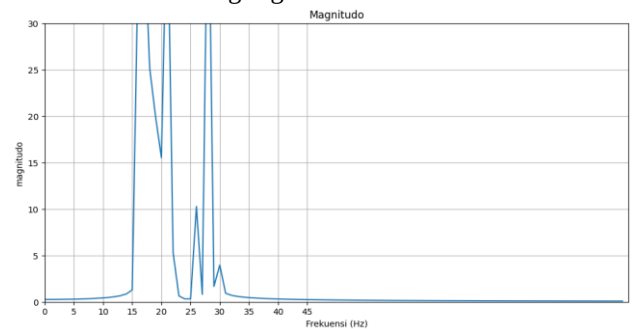


Figure 10. Magnification of the DFT magnitude graph

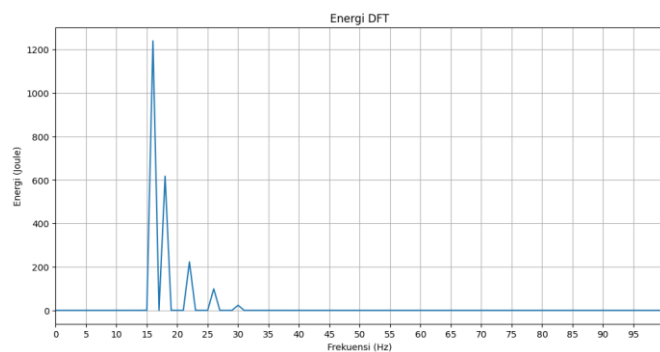


Figure 11. DFT energy graph

Figure 11 shows the signal energy generated by the simulation after DFT. Energy is calculated from the square of the magnitude, which indicates the power of each frequency component of the signal. Peaks on the graph indicate dominant frequencies, which, in the context of EEG, can be associated with specific neurological conditions. However, in this study, because it is limited to a time signal example, it is assumed to be "similar" to the beta waves in Figure 3, namely, a person concentrating for 1 second. Therefore, during this time, the frequency with the highest or dominant energy is the beta wave frequency that makes up its continuous function. If a person is not concentrating and the beta wave activity in the brain has a vast spectrum (magnitude and energy), this condition is abnormal.

IV. CONCLUSION

Based on theoretical analysis and numerical simulations, it can be concluded that the Discrete Fourier Transform (DFT) is a linear mapping that operates in a finite-dimensional complex vector space with a Hilbert space structure. The proof is carried out by ensuring that the space satisfies all the axioms of an inner product space and a complete metric space. Thus, the domain in which the DFT operates can be classified as a finite-dimensional Hilbert space. Furthermore, this study also applies the DFT to investigate mathematically model EEG signals. Using a discretization process that employs the impulse sampling method, the continuous signal is successfully converted into a discrete form suitable for the DFT. The transformation results show that the resulting frequency spectrum matches the frequency components implemented in the initial model, both in terms of magnitude and energy.

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