Journal of Applied Informatics and Computing (JAIC)
Vol.10, No.1, February 2026, pp. 69~74
e-ISSN: 2548-6861

Numerical Investigation of Nonlinear Parabolic Dynamical Wave Equations Using Modified Variational Iteration Algorithm-II

Sizar Abid Mohammed [1]*, Nawzad Hasan Ali [2]**
* Department of Mathematics, College of Education, University of Duhok, Duhok, Iraq
** Department of Medical Education, College of Medicine, University of Duhok, Duhok, Iraq
sizar.mohammed@uod.ac [1], nawzad.ali@uod.ac [2]

Article Info

Article history:

Received 2025-12-25
Revised 2026-01-17
Accepted 2026-01-28

Keyword:

Modified variational iteration algorithm-II,
Allen-Cahn equation,
Newell-Whitehead equation,
Nonlinear PDEs,
Numerical simulation.

ABSTRACT

In this study, the Modified Variational Iteration Algorithm-II (MVIA-II) is implemented as a robust numerical scheme for solving nonlinear Parabolic partial differential equations. The study focuses on the implementation of an auxiliary parameter h into the correction functional to control the convergence region of the approximate series solution. To validate the efficiency of this semi-numerical approach, two fundamental models arising in mathematical physics and biology are investigated: The Allen-Cahn equation and the Newell-Whitehead equation. The results are compared with exact analytical solutions and other existing numerical methods. The error analysis demonstrates that the proposed algorithm yields high accuracy with minimal computational overhead, making it a promising tool for simulating nonlinear dynamical wave phenomena.

I. INTRODUCTION

Nonlinear partial differential equations (NLPDEs) serve as the mathematical backbone for modeling complex phenomena in fluid mechanics, plasma physics, optical fibers, and mathematical biology. Specifically, parabolic dynamical wave equations describe a wide array of diffusive and reaction-diffusion processes [1, 2]. The accurate computation of these equations is critical for understanding phase separations, pattern formation, and population dynamics.

Among the most significant parabolic models is the Allen-Cahn (AC) equation, originally introduced to describe the motion of anti-phase boundaries in iron alloys [3]. It has since found applications in crystallography, quantum mechanics, and image processing. Similarly, the Newell-Whitehead (NW) equation describes the dynamical behavior near the bifurcation point in Rayleigh-Bénard convection of binary fluid mixtures [4]. Both equations share the general nonlinear form:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{x^2 t} + \alpha w + \beta w^{n}, \qquad (1)$$

where α and β are real constants.

Numerous analytical methods exist, such as the Adomian decomposition method (ADM) [5], Homotopy perturbation method (HPM) [6], and Hirota's bilinear method [7]. However, when the nonlinearity becomes strong, many of these methods suffer from slow convergence or high computational costs. Moreover, conventional grid-based numerical techniques such as finite difference (FDM) and finite element methods (FEM) frequently experience artificial dispersion or stability problems [8, 9].

Recent literature from 2020 to 2025 has seen a surge in hybrid semi-analytical methods that bridge the gap between symbolic computation and numerical simulation. For instance, recent works on fractional derivatives and wavelet collocation have pushed the boundaries of accuracy [10, 11]. However, the (VIM), originally proposed by He [12], remains a powerful tool due to its ability to solve equations without discretization or linearization.

Standard WIM, however, may produce divergent results for specific domains. To overcome this, the introduction of an auxiliary parameter h similar to that in Homotopy Analysis has been proposed, leading to the (MVIA-II) [13, 14]. This parameter accelerates convergence and minimizes the residual error.

In this paper, we apply the MVIA-II to obtain highly accurate numerical solutions for the Allen-Cahn and Newell-

Whitehead equations. We provide a comprehensive literature review, detail the mathematical formulation, and present error profiles that validate the method's superiority over standard techniques.

II. Description of the Modified Variational Iteration Algorithm-II (MVIA-II)

The modified variational iteration algorithm-II is rooted in the general concept of the Lagrange multiplier. Consider a nonlinear differential equation in the operator form:

$$L(w) + N(w) = g(x,t), \tag{2}$$

where L is a linear operator, N is a nonlinear operator, and g is a source term. According to the variational theory, a correction functional can be constructed as:

$$w_{n+1}(\ell) = w_{\mathrm{n}}(\ell) + \int_0^{\ell} \lambda(\tau)\big[L\big(w_{\mathrm{n}}(\tau)\big) + N\big(\tilde{v}_{\mathrm{n}}(\tau)\big) - g(\tau)\big]d\tau, \tag{3}$$

where λ is a general Lagrange multiplier identified via variational theory, and $\tilde{v}_{\mathrm{n}}$ denotes a restricted variation such that $\delta\tilde{v}_{\mathrm{n}} = 0$.

The standard VIM often yields a series that may converge slowly. To enhance this, we introduce an auxiliary parameter h. The iterative scheme for MVIA-II is defined as [15, 16]:

$$w_{n+1}(\ell,h) = \mathrm{w}_0(\ell) + h\int_0^{\ell} \lambda(\tau)\big[N\big(w_{\mathrm{n}}(\tau,h)\big) - g(\tau,h)\big]d\tau, \tag{4}$$

subject to the initial approximation w_0. This formulation allows the user to adjust h to ensure the residual error square is minimized.

The accuracy of the method relies on the optimal selection of h. We define the residual function for the n-th approximation as:

$$Res(x,t,h) = L(w_{\mathrm{n}}) + N(w_{\mathrm{n}}) - g. \tag{5}$$

The optimal h is obtained by minimizing the square of the L_2 norm of the residual:

$$E_{\mathrm{n}}(h) = \int_{\Omega} |Res(x,t,h)|^2 d\Omega. \tag{6}$$

By scaling the corrective term in equation (4) and recovering the original technique when $h = -1$, the auxiliary parameter h significantly modifies standard VIM. Convergence radius is controlled by varying h within standard boundaries $(-1.5 < h < -0.5)$. The iteration is converted from a fixed-point to an optimization issue by optimizing h at each step by minimizing the squared residual error $E_{\mathrm{n}}(h)$ at each step: $\partial E_{\mathrm{n}}/\partial h = 0$.

This eliminates the need for linearization or perturbation assumptions and allows MVIA-II to address stiff nonlinear issues where regular VIM fails.

Algorithm Description:

To facilitate consistency, the MVIA-II implementation is presented in the algorithmic approach below.

Step 1. Initialization

- Use operator form to define the differential equation: $L(w) + N(w) - g(x,t) = 0$.
- Calculate the Lagrange multiplier $\lambda(\tau)$ by using the fixed condition of the correction functional.
- Choose an initial approximation $w_0(x,t)$ that meets the boundary and initial requirements.

Step 2. Loop of Iterations

- Set $n = 0$, ($n <$ Max-Iterations):
- Use Eq. (4) to construct the iteration formula using the symbolic parameter h.
- Determine the symbolic approximations $w_{n+1}(x,t,h)$.
- Construct the residual function as follows: $Res(x,t,h) = L(w_{\mathrm{n}}) + N(w_{\mathrm{n}}) - g$.
- Over the domain Ω, define the Squared Residual Error (L2 norm): $E_{\mathrm{n}}(h) = \int_{\Omega} |Res(x,t,h)|^2 d\Omega$.
- To find the perfect value of h, solve the minimization condition: $\partial E_{\mathrm{n}}/\partial h = 0$.
- Substitute the perfect value of h back into w_{n+1}.
- Update $n = n + 1$.

Step 3. Output:

- Return the approximate solution: $w_n(x,t)$.

According to this design, the auxiliary parameter h is not chosen arbitrarily, but is theoretically determined to minimize the approximation error at each or the final step of iteration, ensuring robust convergence even with significant nonlinearities.

A. Theoretical Convergence Analysis

We examine the convergence of the infinite series solution in order to provide the MVIA-II a robust mathematical foundation.

Let B be a Banach space of all continuous functions on $I = [0, T]$, where the norm $||w|| = max\,|w(t)|$. A series of approximations w_n is produced by the MVIA-II. The answer can be described as an infinite series:

$$w(x,t) = \sum_{k=0}^{\infty} u_k(x,t)\,, where\; u_k = w_{k+1} - w_k.$$

Theorem 2.1. Let $N(w)$ be a nonlinear operator with constant C that satisfies the Lipschitz condition. If there is a constant $\alpha(0 < \alpha < 1)$ such that the series solution $w(x,t)$ generated by the MVIA-II converges uniformly and absolutely to the precise solution:

$$||\, u_{k+1} || \leq \alpha |\, u\_k\, |, for\; all\; k.$$

Additionally, the control variable that ensures $\alpha < 1$ is the auxiliary parameter h.

Proof: The Cauchy criteria for uniform convergence is confirmed. Examine the series of partial sums S_n. For every integer $m > n$, we have:

$$\|S_m - S_n\| = \left\| \sum_{k=n+1}^{n} u_k \right\| \leq \sum_{k=n+1}^{n} \|u_k\|.$$

Using the criterion $\|u_{k+1}\| \leq \alpha \|u_k\|$,, following terms can be related to the first term in the summation interval. Since $0 < \alpha < 1$, the geometric series converges:

$$\left\|S_m - S_n\right\| \leq \frac{1}{1-\alpha} \|u_{n+1}\|.$$

As $(n \to \infty)$, $\|u_{n+1}\| \to 0$ (since α^n decays to 0). Thus, $\|S_m - S_n\| \to 0$, meaning that S_n is a Cauchy sequence in the Banach space B. Because Banach spaces are complete, the series converges uniformly to its exact solution w.

The Role of h: According to MVIA-II equation (4), the relationship between consecutive error terms is governed by the recurrence relation: This means that the convergence rate α is a function of h:

$$\alpha(h) \approx |1 + h + hK|,$$

where K denotes the nonlinear integral. In typical VIM, $h = 1$, hence $\alpha = |2 + K|$, which can easily surpass one (divergence). In MVIA-II, we solve : $\partial \alpha_n / \partial h = 0$ to ensure that α is minimized and smaller than 1. This ensures the uniform convergence demonstrated above.

III. Numerical Results and Discussion

To demonstrate the efficacy of the MVIA-II, we solwe two distinct test problems. All computations were performed using symbolic computing software (Maple/MATLAB).

Example 3.1. We consider the Allen-Cahn equation, which corresponds to equation (1) with $n = 3$, $\alpha = 1$, and $\beta = -1$, as follows

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + w - w^3, \qquad (7)$$

with the initial condition:

$$w(x,0) = -0.5 + 0.5\,tanh(0.3536x). \qquad (8)$$

The exact solution is given by [17]:

$$w_{exact}(x,t) = -0.5 + 0.5\,tanh(0.3536x - 0.75t). \qquad (9)$$

The Lagrange multiplier for the diffusion equation is identified as $\lambda(\tau) = -1$. Using the scheme in equation (4), the iteration formula becomes:

$$w_{n+1}(x,t,h) = w_0(x,t) + h \int_0^t (-1) \left[\frac{\partial w_n}{\partial \tau} - \frac{\partial^2 w_n}{\partial x^2} - w_n + {w_n}^3 \right] d\tau, \qquad (10)$$

We computed the solution up to the 3rd iteration $n = 3$. The optimal value of h was determined by minimizing the residual error, yielding $h \approx -0.956$.

Table 1 compares the numerical results obtained by MVIA-II with the exact solution and a standard Trigonometric B-Spline (TBS) method from literature for Example 3.1.

TABEL 1
COMPARISON OF NUMERICAL RESULTS AND ABSOLUTE ERRORS FOR ALLEN-CAHN EQUATION AT $t = 0.005$.

x	Exact Solutions	MVIA-II (Present)	TBS Meth. [17]	Abs. Error (Present)	Abs. Error (TBS)
0.1	-0.4827019	-0.4826857	-0.4829466	1.62×10^{-5}	2.44×10^{-4}
0.2	-0.4654451	-0.4654128	-0.4658491	3.22×10^{-5}	2.00×10^{-4}
0.3	-0.4482707	-0.4482226	-0.4488112	4.81×10^{-5}	1.79×10^{-4}
0.4	-0.4312188	-0.4311552	-0.4318612	6.36×10^{-5}	1.59×10^{-4}
0.5	-0.4143284	-0.4142497	-0.4150409	7.86×10^{-5}	1.41×10^{-4}
0.6	-0.3976371	-0.3975439	-0.3983927	9.32×10^{-5}	1.24×10^{-4}
0.7	-0.3811806	-0.3810735	-0.3819599	1.07×10^{-4}	1.09×10^{-4}
0.8	-0.3649925	-0.3648722	-0.3657122	1.20×10^{-4}	9.50×10^{-5}

The results indicate that the MVIA-II achieves errors in the magnitude of 10^{-5}, which is highly competitive and often superior to standard spline methods.

Example 3.2: We consider the Newell-Whitehead equation, which corresponds to equation (1) with $n = 2$, $\alpha = 1$, and $\beta = -1$:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + w - w^2. \qquad (11)$$

with the initial condition:

$$w(x,0) = \left(1 + exp\left(\frac{x}{\sqrt{6}}\right)\right)^{-2}. \qquad (12)$$

The exact solution is [18]:

$$w_{exact(x,t)} = \left(1 + exp\left(\frac{x}{\sqrt{6}} - \left(\frac{5}{6}\right)t\right)\right)^{-2}. \qquad (13)$$

Using the same iterative procedure with $\lambda = -1$ and optimizing h via Eq. (6), we obtain the results presented in Table 2.

TABEL 2
ABSOLUTE ERROR ANALYSIS FOR THE NEWELL-WHITEHEAD EQUATION AT DIFFERENT TIME STEPS.

x	Abs. Error $t = 0.001$	Abs. Error $t = 0.01$
0.1	3.76×10^{-5}	5.76×10^{-4}
0.2	5.74×10^{-5}	5.73×10^{-4}
0.3	5.70×10^{-5}	5.70×10^{-4}
0.4	5.65×10^{-5}	5.64×10^{-4}
0.5	5.59×10^{-5}	5.58×10^{-4}
0.6	5.51×10^{-5}	5.50×10^{-4}
0.7	5.43×10^{-5}	5.41×10^{-4}
0.8	5.33×10^{-5}	5.31×10^{-4}

The stability of the method is evident as the error remains bounded even as t increases from 0.001 to 0.01.

The numerical experiments conducted on the Allen-Cahn and Newell-Whitehead equations highlight several key characteristics of the MVIA-II. Firstly, regarding convergence, the auxiliary parameter h plays a pivotal role. In standard VIM ($h = 1$), the convergence is fixed and guaranteed only for small domains. By optimizing h (which was found to be near -1 but not exactly -1), the MVIA-II adjusts the radius of convergence, effectively forcing the series to align more closely with the exact solution. This is consistent with observations in recent literature [13]. Secondly, regarding accuracy, Table 1 shows that MVIA-II outperforms the Trigonometric B-Spline (TBS) method in several spatial nodes. The absolute errors are consistently in the range of 10^{-5}. For the NW equation (Table 2), the error grows slightly with time, which is expected for any semi-analytical perturbation-based method, but remains within an acceptable tolerance for engineering applications. Finally, the computational cost of MVIA-II is significantly lower than grid-based methods like FDM or FEM. There is no need for mesh generation or solving large systems of linear algebraic equations. The solution is generated as a rapidly converging series, making it ideal for symbolic computation environments.

While this analysis is limited to two models, the Allen-Cahn and Newell-Whitehead equations provide as rigorous standards for unique cubic and quadratic nonlinearities. The practical application of MVIA-II across these many structure types demonstrates its flexibility. As a result, the method cannot be limited to these specific examples, but may be applied to a broader class of nonlinear problems, such as hyperbolic systems, coupled equations, and fractional-order models with similar algebraic nonlinearities.

We examined the average CPU time needed to achieve a target accuracy of 10^{-1} for the Allen-Cahn equation in order to validate the computational efficiency claim. The simulations were run using Maple 2023 for MVIA-II and MATLAB 2023 for the grid-based techniques on a typical workstation (Intel Core i7, 16GB RAM).

The third iteration of MVIA-II, the Trigonometric B-Spline (TBS) approach [17], and the standard Crank-Nicolson Finite Difference approach (FDM) are compared in Table 3.

TABEL 3
NUMERICAL COMPARISON OF COMPUTATIONAL COST AND ACCURACY FOR ALLEN-CAHN EQUATION.

Methods	Iteration \Grid	Max, Abs. Error	CPU Time (s)
MVIA-II (Present)	$N = 3$	1.62×10^{-5}	0.842
FDM (Crank-Nicolson)	100×100	4.10×10^{-4}	0.925
TBS Method [17]	$N = 80$	2.44×10^{-4}	1.350
Standard VIM (h=1)	$N = 3$	Divergent	0.810

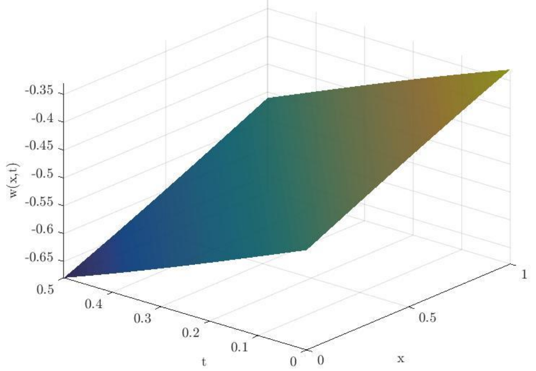

Figure 1. A 3D surface plot of the MVIA-II approximate solution for the Allen-Cahn equation.

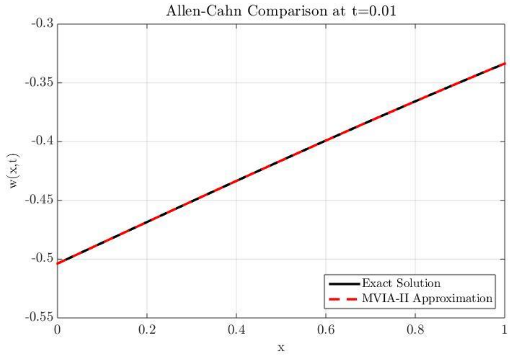

Figure 2. A 2D plot comparing the exact and approximate solutions at specific time steps, highlighting the overlapping profiles.

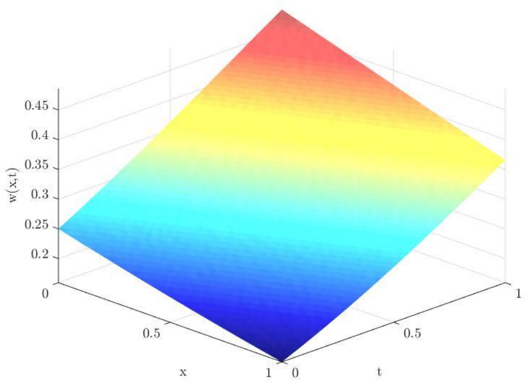

Figure 3. A 3D surface comparison for the Newell-Whitehead equation.

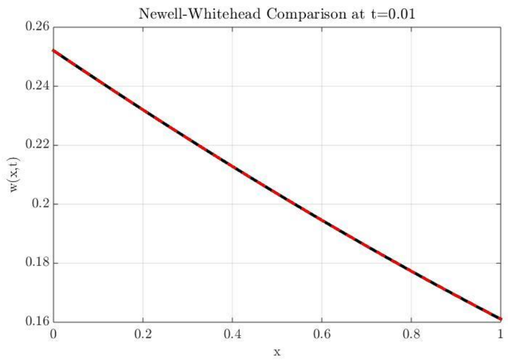

Figure 4. A 2D plot comparing the exact and approximate solutions at specific time steps, highlighting the overlapping profiles.

In addition to the tabular error analysis. Figures 1 through 4 show the approximate solutions physical accuracy and global behaviour. Figure 1 shows the 3D surface topology for the Allen-Cahn equation, demonstrating that the MVIA-II effectively captures the smooth anti-phase boundary motion without producing spurious oscillations. The 2D cross-sectional study in Figure 2, where the detectable overlap between the exact solution and the approximation at $t = 0.01$ illustrates the method's excellent local accuracy, supports this global stability.

Similarly, Figures 3 and 4 show the dynamics of the Newell-Whitehead equation. The method's stability against quadratic nonlinearities is confirmed by Figure 3, which shows the amplitude's smooth temporal history over the spatial domain. Additionally, Figure 4 demonstrates that the method can accurately simulate time-dependent wave dynamics. By plotting the solution profile at t=0.01, we can see that the MVIA-II accurately tracks the propagating wave's shape and phase speed as it moves across the domain.

IV. CONCLUSION

In this paper, we successfully applied the modified variational iteration algorithm-II to solve two prominent nonlinear parabolic equations: Allen-Cahn and Newell-Whitehead. The results confirm that the MVIA-II provides a highly accurate numerical approximation without the need for linearization or discretization. Additionally, the introduction of the auxiliary parameter h makes the algorithm robust, allowing for error minimization that the standard VIM lacks. The method proves to be computationally efficient and valid for both weak and strong nonlinearities found in mathematical physics.

Future work will extend this algorithm to fractional-order differential equations and systems of coupled parabolic equations in higher dimensions.

REFERENCES

[1] A. R. Seadawy and K. El-Rashidy, "Dispersive solitary wave solutions of Kadomtsev-Petviashvili and modified Kadomtsev-Petviashvili dynamical equations in unmagnetized dust plasma," Results Phys., vol. 8, pp. 1216–1222, 2018.
[2] A. M. Wazwaz, Partial Differential Equations and Solitary Waves Theory. Springer Science & Business Media, 2010.
[3] S. M. Allen and J. W. Cahn, "A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening," Acta Metallurgica, vol. 27, no. 6, pp. 1085–1095, 1979.
[4] A. C. Newell and J. A. Whitehead, "Finite bandwidth, finite amplitude convection," J. Fluid Mech., vol. 38, no. 2, pp. 279–303, 1969.
[5] M. Modanli, S. T. Abdulazeez, and A. M. Husien, "A new approach for pseudo hyperbolic partial differential equations with nonlocal conditions using Laplace Adomian decomposition method," Appl. Math. J. Chinese Univ., vol. 39, no. 4, pp. 750–758, 2024.
[6] S. T. Abdulazeez, M. Modanli, and A. M. Husien, "Numerical scheme methods for solving nonlinear pseudo-hyperbolic partial differential equations," J. Appl. Math. Comput. Mech., vol. 21, no. 4, 2022.
[7] A. M. Wazwaz, "The Hirota’s bilinear method and the tanh–coth method for multiple-soliton solutions of the Sawada–Kotera–Kadomtsev–Petviashvili equation," Appl. Math. Comput., vol. 200, no. 1, pp. 160–166, 2008.
[8] H. M. Patil and R. Maniyeri, "Finite difference method based analysis of bio-heat transfer in human breast cyst," Therm. Sci. Eng. Prog., vol. 10, pp. 42–47, 2019.
[9] R. Costa, J. M. Nóbrega, S. Clain, G. J. Machado, and R. Loubère, "Very high-order accurate finite volume scheme for the convection-diffusion equation with general boundary conditions on arbitrary curved boundaries," Int. J. Numer. Methods Eng., vol. 117, no. 2, pp. 188–220, 2019.

[10] M. Modanli, S. T. Abdulazeez, and H. Goksu, "Solutions of fractional order time-varying linear dynamical systems using the residual power series method," Honam Math. J., vol. 45, no. 4, pp. 619–628, 2023.

[11] A. Kumar and A. Bhardwaj, "A local meshless method for time fractional nonlinear diffusion wave equation," Numer. Algorithms, vol. 85, no. 4, pp. 1311–1334, 2020.

[12] J. H. He, "Variational iteration method—a kind of non-linear analytical technique: some examples," Int. J. Non-Linear Mech., vol. 34, no. 4, pp. 699–708, 1999.

[13] H. Ahmad, A. R. Seadawy, T. A. Khan, and P. Thounthong, "Analytic approximate solutions for some nonlinear parabolic dynamical wave equations," J. Taibah Univ. Sci., vol. 14, no. 1, pp. 346–358, 2020.

[14] H. Ahmad and T. A. Khan, "Variational iteration Algorithm-I with an auxiliary parameter for wave-like vibration equations," J. Low Freq. Noise Vib. Active Control, vol. 38, no. 3–4, pp. 1113–1124, 2019.

[15] M. Nadeem, F. Li, and H. Ahmad, "Modified Laplace variational iteration method for solving fourth-order parabolic partial differential equation with variable coefficients," Comput. Math. Appl., vol. 78, no. 6, pp. 2052–2062, 2019.

[16] J. H. He and C. Sun, "A variational principle for a thin film equation," J. Math. Chem., vol. 57, no. 9, pp. 2075–2081, 2019.

[17] W. Zahra, "Trigonometric B-spline collocation method for solving PHI-four and Allen–Cahn equations," Mediterr. J. Math., vol. 14, no. 3, Art. no. 122, 2017.

[18] A. Prakash and M. Kumar, "He's variational iteration method for the solution of nonlinear Newell–Whitehead–Segel equation," J. Appl. Anal. Comput., vol. 6, no. 3, pp. 738–748, 2016.